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Original article

Generalized Semi-α-Closed Sets in Topology

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Abstract
In 1969, Levine introduced the concept and properties of generalized closed sets (briefly g-closed), where the complement of such a set is called a generalized open set (briefly g-open). In this research, we introduce and study new classes of sets called generalized semi-α-closed sets (briefly gsα-closed) in topological spaces. We investigate and prove their relationships with other closed sets, supported by examples and counterexamples, and establish their fundamental properties such as union, intersection, and containment. We also present definitions for the closure of generalized semi-α-closed sets (briefly, $\text{gs}\alpha-\text{cl}(A)$) and the interior of generalized semi-α-closed sets (briefly, $\text{gs}\alpha-\text{int}(A)$), studying their key properties, providing illustrative examples, and proving their fundamental characteristics. In future studies, we aim to expand this research by introducing a new operator similar to the one currently studied in terms of topological properties.
Keywords. g- closed, s- closed, α- closed gsα- closed, gsα- open, gsα-cl(A), gsα-int(A).

Introduction
In 1970, Levine [1] first proposed the notion of generalized closed set (g-closed). These sets have since uncovered significant new properties in topological spaces, inspiring extensive research by numerous scholars in subsequent years. NJasted [2] initially explored the concept of α- sets, which were later termed as α-open sets in 1983. Mashhours et. al [3] further investigated α- closed sets, α-closure of a set-in topological spaces. Additionally, Maki et. al [4, 5] contributed to the field by examining generalized α-closed sets and α- generalized closed sets. This paper seeks to advance the study of generalized semi α- closed sets (briefly, gs α-closed) by introducing novel insights and concepts through analytical and research-driven methodologies. It presents the definition of generalized semi α- closed sets (briefly, gs α-closed) and establishes various characterizations of these sets. Unless specified otherwise, (X,τ) denotes a non-empty topological space without any assumed separation axioms. The present research centers on generalized semi α- closed sets (briefly, gs α-closed) and investigates their fundamental properties within a topological space. By analyzing these sets and their features, this study aims to enhance the understanding of their significance in topology.

Preliminaries
Throughout this work, we denote a topological space by X or (X,τ). Unless otherwise specified, no separation axioms are assumed. The following definitions will be used in subsequent sections.

Definition 2.1.
Let (X,τ) be a topological space, and let A is a non-empty subset of X [6]:
1- The closure of A, denoted $cl(A)$, is the intersection of all closed sets containing A.
2- The interior of A denoted $int(A)$, is the union of all open sets contained in A.

Definition 2.2.
Let (X,τ) be a topological space, and let A be a non-empty subset of X is called:
1- Semi –closed set [7] if $\text{int}(\text{cl}(A)) \subseteq A$ and semi-open set if $A \subseteq \text{cl}(\text{int}(A))$.
2- $\alpha-$closed set [2] if $\text{cl}\left(\text{int}(\text{cl}(A))\right) \subseteq A$ and $\alpha-$open set if $A \subseteq \text{int}\left(\text{cl}(\text{int}(A))\right)$.
3- Regular- closet set (briefly r-closed) [8] if $A = \text{cl}(\text{int}(A))$ and regular- open set if $A = \text{int}(\text{cl}(A))$.

Definitition 2.3
Let (X,τ) be a topological space. A subset A of (X,τ) is called:
1- Generalized closed set (briefly g-closed) [1,9] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
2- Semi-generalized closed set (briefly sg-closed) [10] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
3- Generalized semi-closed set (briefly gs-closed) [11,12] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
4- $\alpha-$generalized closed set (briefly αg- closed) [5] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
5- Generalized α-closed set (briefly gα- closed) [4, 13] if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is α- open in X.

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6- Regular generalized closed set (briefly rg- closed) [14] if $\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is r-open in X.

Remark 2.4

For any subset U of a topological space X [15], the following hierarchy relations hold:

1- Closed- type sets:
- Regular closed set $\Rightarrow$ closed set $\Rightarrow$ semi closed set.
- Regular closed set $\Rightarrow$ closed set $\Rightarrow$ g- closed set.

2- Open- type sets:
- Regular open set $\Longrightarrow$ open set $\Rightarrow$ semi open set.
- Regular open set $\Rightarrow$ open set $\Rightarrow$ g- open set.

Remark 2.5.

For any subset A of X,[15] the following containment relations hold:

1- $\mathrm{scl}(A) \subseteq \mathrm{cl}(A) \subseteq \mathrm{rcl}(A)$.
2- $\mathrm{gcl}(A) \subseteq \mathrm{cl}(A) \subseteq \mathrm{rcl}(A)$.

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In this section, we introduce the concept of generalized Semi α- Closed sets (briefly gsα- closed) in a topological space and investigate their fundamental properties.

Definition 3.1.

Let (X, τ) be a topological space. A subset $A \subseteq X$ is called:

1- generalized semi α- closed (briefly gsα- closed) if $\mathrm{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is α- open in X.
2- Generalized semi α- open (briefly gsα- open) if its complement is generalized semi α- closed set (briefly gsα- closed). The collection of all generalized semi α- closed sets (briefly, gsα- closed) in X is denoted by $(\mathrm{Gs\alpha C}(X))$.

Example 3.2

Consider the topological space (X, τ) where $X = \{a, b, c\}$ with topological $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. The collection of generalized semi α- closed sets (briefly gsα- closed) in X is:
$(\mathrm{Gs\alpha C}(X)) = \{X, \phi, \{a\}, \{b\}\}$.

Remark 3.3

From definition 3.1 we observe the following inclusion relation:
- Every generalized semi α- closed sets (briefly, gsα- closed) is semi- closed set (briefly s-closed).
- The converse does not hold in general.

Example 3.4

In Example 3.2. The subset $\{a, c\}$ is semi- closed set (briefly s-closed) but not generalized semi α- closed sets (briefly gsα- closed).

Remark 3.5

Definition 3.1 establishes the following relationship
- Every generalized semi α- closed sets (briefly, gsα- closed) is generalized semi- closed set (briefly gs- closed).
- The converse is not necessarily true.

Example 3.6

In Example 3.2. The subset $\{a, b\}$ is a generalized semi-closed set (briefly gs-closed) but not generalized semi α- closed sets (briefly, gsα- closed).

Remark 3.7

Through Definition 3.1. and example 3.2, we establish the following independence results:
- Closed sets and generalized semi α- closed sets (briefly gsα- closed) are independent concepts.
- Semi-generalized- closed set (briefly sg - closed) and generalized semi α- closed sets (briefly gsα- closed) are independent concepts.

Example 3.8

From Example 3.2. We observe:

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- The subset $\{c\}$ is closed set but is not generalized semi α- closed sets (briefly gsα- closed). Conversely, $\{b\}$ is generalized semi α- closed sets (briefly gsα- closed) but not closed set.
- The subset $\{c\}$ is semi generalized- closed set (briefly sg - closed) but not generalized semi α- closed sets (briefly gsα- closed). While $\{X\}$ is generalized semi α- closed sets (briefly gsα- closed) but not semi-generalized- closed set (briefly sgα - closed).

Remark 3.9.

Further analysis reveals:

- α- closed set (briefly α - closed) and generalized semi α- closed sets (briefly gsα- closed) are independent.
- Generalizedα- closed set (briefly gα - closed) and generalized semi α- closed sets (briefly gsα- closed) are independent.

Example 3.10.

From the topological space (X, τ) in Example 3.2. we observe:

- The subset $\{a, c\}$ is α- closed set (briefly α - closed) set but not generalized semi α- closed sets (briefly gsα- closed). While $\{b\}$ is generalized semi α- closed sets (briefly gsα- closed) but not α- closed set (briefly α - closed).
- The subset $\{a, b\}$ is generalized α- closed set (briefly gα - closed) but not generalized semi α- closed sets (briefly gsα- closed).While $\{a\}$ is generalized semi α- closed sets (briefly gsα- closed) but not α- closed set (briefly α - closed).

Theorem 3.11.

The union of two generalized semi α- closed sets (briefly gsα- closed) is generalized semi α- closed set (briefly gsα- closed).

Proof

Let $A, B \in Gs\alpha C(X)$ and let U be an α- open set (briefly α - open) in X containing $A \cup B$, such that $A \subseteq U$ and $B \subseteq U$. Since A and B are generalized semi α- closed sets (briefly gsα- closed), then $\text{scl}(A) \subseteq U$ and $\text{scl}(B) \subseteq U$. But by properties of semi –closure, $\text{scl}(A \cup B) = \text{scl}(A) \cup \text{scl}(B) \subseteq U$. Therefore, $A \cup B$ is generalized semi α- closed sets (briefly gsα- closed).

Theorem 3.12.

If A is generalized semi α- closed sets (briefly gsα- closed) in X and $A \subseteq B \subseteq \text{scl}(A)$, then B is also generalized semi α- closed sets (briefly gsα- closed).

Proof

Let U be an α- open set in X, such that $B \subseteq U$,since $A \subseteq B$ we have $A \subseteq U$. Because A is generalized semi α- closed sets (briefly gsα- closed), $\text{scl}(A) \subseteq U$. By hypothesis, $B \subseteq \text{scl}(A)$, so $\text{scl}(B) \subseteq \text{scl}(A)$. Therefore, $\text{scl}(B) \subseteq U$. Thus, B is generalized semi α- closed sets (briefly gsα- closed).

Theorem 3.13.

If A is both α- open set and generalized semi α- closed sets (briefly gsα- closed) in X, then A is semi closed (s- closed).

Proof

Since $A \subseteq A$, and A is α- open set. The generalized semi α- closed sets (briefly gsα- closed) property implies $\text{scl}(A) \subseteq A$. However, $A \subseteq \text{scl}(A)$ always holds. Thus $A = \text{scl}(A)$, meaning A is semi closed (s- closed).

Theorem 3.14.

Let $A \subseteq Y \subseteq X$ where A is generalized semi α- closed sets (briefly gsα- closed) in X. Then A is generalized semi α- closed sets (briefly gsα- closed) relative to Y.

Proof

Suppose $A \subseteq Y \cap G$, where G is α- open set in X. Since $A \subseteq G$ and A is generalized semi α- closed sets (briefly, gsα- closed) in X, we have $\text{scl}(A) \subseteq G$. That is $Y \cap \text{scl}(A) \subseteq Y \cap G$, where $Y \cap \text{scl}(A)$ is the semi- closure of A. Thus A is generalized semi α- closed sets (briefly gsα- closed) relative to Y.

Generalized semi α- closure and generalized semi α- interior.

In this section, we introduce the concepts of generalized semi α- Closure (briefly gsα- closure) and generalized semi α- interior (briefly gsα- interior) for a subset A of X, using generalized semi α- closed sets (briefly, gsα- closed). We also investigate their fundamental properties.

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Definition 4.1.

For a subset A of X, the Generalized semi α- Closure (briefly gsα- closure) of A,denoted by gs$\alpha - \text{cl}(A)$, is defined as the the intersection of all generalized semi α- closed sets (briefly gsα- closed) containing A That is:

$$\text{gs}\alpha - \text{cl}(A) = \cap \{G: A \subseteq G, G \text{ is gs}\alpha - \text{closed in X}\}.$$

Definition 4.2.

For a subset A of X, the generalized semi α- interior (briefly gsα- interior) of A, denoted by gsα- int (A), is defined as the union of all all generalized semi α- open sets (briefly gsα- open) contained in A That is:

$$\text{gs}\alpha\text{- int }(A) = \cup \{G: G \subseteq A, G \text{ is gs}\alpha - \text{open in } X\}.$$

Remark 4.3.

The following example demonstrates that the inclusions

$A \subseteq \text{gs}\alpha - \text{cl(A)} \subseteq \text{cl}(A)$ and $\text{int}(A) \subseteq \text{gs}\alpha - \text{int}(A) \subseteq A$.

Do not hold in general.

Example 4.4.

Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ be a topology on X.

- For subsets of X, $A = \{b\} \subseteq \text{gs}\alpha - \text{cl(A)} = \{\text{a, b}\}$ but not $\subseteq \text{cl}(A) = \{\text{b}\}$.
- For subsets of X, $A = \{a, d\}$ then, $\text{int}(A) = \{a\} \subseteq \text{gs}\alpha - \text{int(A)} = \{\text{a, b}\}$ but not $\subseteq A = \{\text{a, d}\}$.

Theorem 4.5.

If A and B are subsets of X, then:

1- $\text{gs}\alpha - \text{cl}(X) = X$ and $\text{gs}\alpha - \text{cl}(\phi) = \phi$.
2- $A \subseteq \text{gs}\alpha - \text{cl(A)}$.
3- If B is any gs$\alpha -$ closed set containing A, then $\text{gs}\alpha - \text{cl(A)} \subseteq B$.
4- If $A \subseteq B$, then $\text{gs}\alpha - \text{cl}(A) \subseteq \text{gs}\alpha - \text{cl}(B)$.
5- $\text{gs}\alpha - \text{cl}(A) = \text{gs}\alpha - \text{cl}\left(\text{gs}\alpha - \text{cl}(A)\right)$.

Proof

(1), (2), (3) and (4) fellow directly from Definition 4.1.

(5) Let D be gs$\alpha -$ closed set containing A. By definition 4.1, $\text{gs}\alpha - \text{cl}(A) \subseteq D$. Since D is gs$\alpha -$ closed set containing $\text{gs}\alpha - \text{cl}(A)$, and $\text{gs}\alpha - \text{cl}(A)$ is the smallest such set, it follows that:

$\text{gs}\alpha - \text{cl}\left(\text{gs}\alpha - \text{cl}(A)\right) \subseteq \text{gs}\alpha - \text{cl}(A)$.

Conversely, $\text{gs}\alpha - \text{cl}(A)$ is itself a gs$\alpha -$ closed set containing A, so:

$\text{gs}\alpha - \text{cl}(A) \subseteq \text{gs}\alpha - \text{cl}\left(\text{gs}\alpha - \text{cl}(A)\right)$.

Hence $\text{gs}\alpha - \text{cl}(A) = \text{gs}\alpha - \text{cl}\left(\text{gs}\alpha - \text{cl}(A)\right)$.

Theorem 4.6.

Suppose A and B are subsets of X, then:

$$\text{gs}\alpha - \text{cl}(A \cap B) \subseteq \text{gs}\alpha - \text{cl}(A) \cap \text{gs}\alpha - \text{cl}(B).$$

Proof

Since $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$, by theorem 4.5, we have: $\text{gs}\alpha - \text{cl}(A \cap B) \subseteq \text{gs}\alpha - \text{cl}(A)$ and $\text{gs}\alpha - \text{cl}(A \cap B) \subseteq \text{gs}\alpha - \text{cl}(B)$ implies that $\text{gs}\alpha - (\text{A} \cap \text{B}) \subseteq \text{gs}\alpha - \text{cl}(A) \cap \text{gs}\alpha - \text{cl}(B)$.

Remark 4.7.

The equality of Theorem .4.6. does not hold in general as demonstrated by the following example .

Example 4.9.

In example 4.4, the subsets of X, $A = \{a\}$ and $B = \{b\}$. The $\text{gs}\alpha - \text{cl}(A) = \{a, b\}$ and $\text{gs}\alpha - \text{cl}(B) = \{a, b\}$, then $\text{gs}\alpha - \text{cl}(A) \cap \text{gs}\alpha - \text{cl}(B) = \{a, b\}$. But $\text{gs}\alpha - \text{cl}(A \cap B) = \emptyset$. Hence:

$$\text{gs}\alpha - \text{cl}(A \cap B) \not\supseteq \text{gs}\alpha - \text{cl}(A) \cap \text{gs}\alpha - \text{cl}(B)$$

Theorem 4.10.

Suppose that A and B are subsets of X, then:

$$\text{gs}\alpha - \text{cl}(A \cup B) \supseteq \text{gs}\alpha - \text{cl}(A) \cup \text{gs}\alpha - \text{cl}(B).$$

Remark 4.11.

The equality of Theorem .4.7. does not hold in general, as demonstrated by the following example.

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Example 4.12.

In example 4.4, the subsets of X, $A=\{a,b\}$ and $B=\{c\}$. The $\text{gs}\alpha-\text{cl}(A)=\{a,b\}$ and $\text{gs}\alpha-\text{cl}(B)=\{c\}$, then $\text{gs}\alpha-\text{cl}(A\cup B)=X$. Hence

$$\text{gs}\alpha-\text{cl}(A\cup B)\nsubseteq \text{gs}\alpha-\text{cl}(A)\cup\ \text{gs}\alpha-\text{cl}(B).$$

Remark 4.13.

If $A\subseteq X$ and A is $\text{gs}\alpha-$closed set, then $\text{gs}\alpha-\text{cl}(A)$ is the smallest $\text{gs}\alpha-$closed subset of X containing A.

Theorem 4.14.

Suppose that A and B are subsets of X, then:

1- $\text{gs}\alpha-\text{int}(X)=X$ and $\text{gs}\alpha-\text{int}(\phi)=\phi$.

2-If B is any $\text{gs}\alpha-$open set contained in A, then $B\subseteq \text{gs}\alpha-\text{int}(A)$.

Proof

1- follows directly from the definition 4.2.

2- Suppose B is any $\text{gs}\alpha-$open set contained in A. For any $x\in B$, since B is $\text{gs}\alpha-$open set contained in A. Then $x\in\ \text{gs}\alpha-\text{int}(A)$. Thus, $B\subseteq\ \text{gs}\alpha-\text{int}(A)$.

Remark 4.15.

The inclusion $\text{int}(A)\subseteq\ \text{gs}\alpha-\text{int}(A)$ does not hold in general.

Example 4.16.

In example 4.4, the subsets of X, let $A=\{a,c,d\}$ then:
$\text{int}(A)=\{a,c,d\}$ and $\text{gs}\alpha-\text{int}(A)=\{a,d\}$ hence**,** $\text{int}(A)\nsubseteq\ \text{gs}\alpha-\text{int}(A)$.

Conclusion

This paper introduces and examines the concepts of generalized Semi α Closed sets (briefly gsα- closed) and generalized semi α- closure and generalized semi α- interior within topological spaces, exploring their fundamental properties. The generalized Semi α Closed sets (briefly gsα- closed) can be used to derive a new homeomorphisms, connectedness, compactness, and new separation axioms. This concept can be extended to bitopological and fuzzy topological spaces.

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