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Original article

Comparison of Runge-Kutta Methods for Solving Nonlinear Equations

Safaa Abu-Amr[1]*, Tebra Almajbri[2], Mnal Ali[1]

[1]*Department of Mathematics, Faculty of Science, University of Omar Al-Mokhtar, El-Baida, Libya*
[2]*Department of Materials Science and Engineering, Mathematics major, Faculty of Engineering, University of Omar Al-Mokhtar, El-Baida, Libya*

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Corresponding Email. Safaa.soliman@omu.edu.ly

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ABSTRACT

This research explores the numerical solutions of nonlinear ordinary differential equations using four distinct methods: Heun method, the Midpoint method, Ralston's method, and the fourth-order Runge-Kutta method (RK4). Given the complexity and prevalence of nonlinear equations in various scientific fields, effective numerical techniques are essential for obtaining accurate solutions. This research contributes to the understanding of numerical methods in solving complex differential equations, aiding practitioners in selecting the most appropriate approach for their needs.

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INTRODUCTION

Ordinary nonlinear differential equations are common in various fields such as physics, engineering, and biology, as they reflect complex dynamic systems for which traditional analytical solutions are often difficult to find. Therefore, numerical methods have become essential tools for approximating solutions to these equations. This research focuses on four main numerical techniques: the Heun method, the midpoint method, the Ralston method, and the fourth-order Runge-Kutta method (RK4) [1-8].

Heun's Method

The Heun method is a second-order technique within the Runge-Kutta methods and is considered an improvement over the Euler method. This method relies on making an initial prediction based on the first slope, which is then refined using the second slope calculated at the predicted point. This two-step process enhances accuracy while maintaining relatively low computational effort [15,22].

Midpoint Method

The midpoint method offers second-order accuracy by evaluating the function at the midpoint of the interval. This method is characterized by its ability to capture the dynamics of the solution more effectively compared to simple methods, contributing to improved estimates while maintaining ease of implementation [9,23].

Ralston's Method

The Ralston method is a second-order technique that relies on using specific weights for the slope, achieving a balance between accuracy and computational costs. This method calculates two slopes and then combines them to obtain an updated solution, making it suitable for problems that require a moderate level of accuracy [17-19].

Fourth-Order Runge-Kutta Method (RK4)

The RK4 method is widely recognized for its high accuracy in solving ordinary differential equations. This method relies on calculating four slopes and then combining them using specific coefficients to achieve fourth-order accuracy. It is particularly effective in dealing with complex equations, providing a robust framework for obtaining precise solutions over extended time intervals. [14,20,21].

Definition:

The differential equation of the form [11,12,16]

$$\frac{dy}{dx} + P(x)y = Q(x)\, y^n \quad (1)$$

is called a Bernoulli differential equation.

Note: It is important to mention that when $n = 0$ or 1, the Bernoulli equation becomes a linear equation. In fact, we can transform a Bernoulli-type differential equation into a linear differential equation using the following method.

Theorem. Suppose $n \neq 0$ and $n \neq 1$. Then the transition $v = y^{1-n}$ He seeks to simplify the Bernoulli differential equation. $\frac{dy}{dx} + P(x)y = Q(x)\, y^n$ To a linear equation related to v. Notice that if $v = y^{1-n}$ then

$$dv/dx = (1-n)\, y^{-n}\, dy/dx \quad (2)$$

ANALYSIS OF METHODS

Heun Method: [15,20]

The Heun method is an explicit numerical technique for solving first-order ordinary differential equations. This method is considered simple and consists of two steps, as it improves upon the Euler method by taking the average of the slopes:

Start from the initial point (t_0, y_0)

Calculate the slope at the current point $k_1 = f(t_n, y_n)$ (3)

Estimate the next point $k_2 = f(t_n + h, y_n + hk_1)$ (4)

Update y using the average of the slopes

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \quad (5)$$

Midpoint Method: [9,10, 21]

The Midpoint method, also known as the improved Euler method, is another second-order accurate method for solving ordinary differential equations:

Start from (t_0, y_0)

Calculate the slope at the current point $k_1 = f(t_n, y_n)$ (6)

Estimate y at the midpoint $y_{mid} = y_n + \frac{h}{2}k_1$ (7)

Calculate the slope at the midpoint $k_2 = f(t_n + \frac{h}{2}, y_{mid})$ (8)

Update y using the slope at the midpoint

$$y_{n+1} = y_n + hk_2 \quad (9)$$

Ralston's Method [13,18]
Ralston's method is another second-order method that uses a weighted average of two different slopes to achieve higher accuracy:

Start with $(\boldsymbol{t_0}, \boldsymbol{y_0})$

Compute $$\boldsymbol{k_1} = \boldsymbol{f}(\boldsymbol{t_n}, \boldsymbol{y_n}) \quad (10)$$

$$\boldsymbol{k_2} = \boldsymbol{f}\left(\boldsymbol{t_n} + \frac{3\boldsymbol{h}}{4}, \boldsymbol{y_n} + \frac{3}{4}\boldsymbol{h}\boldsymbol{k_1}\right) \quad (11)$$

Update $\boldsymbol{y}$ using a weighted combination of $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$

$$\boldsymbol{y_{n+1}} = \boldsymbol{y_n} + \frac{\boldsymbol{h}}{3}(\boldsymbol{k_1} + 2\boldsymbol{k_2}) \quad (12)$$

Fourth-Order Runge-Kutta Method [19, 20, 21]
The fourth-order Runge-Kutta (RK4) method is one of the most widely used and accurate techniques for solving ordinary differential equations:

Start from $(\boldsymbol{t_0}, \boldsymbol{y_0})$

Calculate the four slopes

$$\boldsymbol{k_1} = \boldsymbol{f}(\boldsymbol{t_n}, \boldsymbol{y_n}) \quad (13)$$

$$\boldsymbol{k_2} = \boldsymbol{f}\left(\boldsymbol{t_n} + \frac{\boldsymbol{h}}{2}, \boldsymbol{y_n} + \frac{\boldsymbol{h}}{2}\boldsymbol{k_1}\right) \quad (14)$$

$$\boldsymbol{k_3} = \boldsymbol{f}\left(\boldsymbol{t_n} + \frac{\boldsymbol{h}}{2}, \boldsymbol{y_n} + \frac{\boldsymbol{h}}{2}\boldsymbol{k_2}\right) \quad (15)$$

$$\boldsymbol{k_4} = \boldsymbol{f}(\boldsymbol{t_n} + \boldsymbol{h}, \boldsymbol{y_n} + \boldsymbol{h}\boldsymbol{k_3}) \quad (16)$$

Update $\boldsymbol{y}$ using a weighted average of all four slopes

$$\boldsymbol{y_{n+1}} = \boldsymbol{y_n} + \frac{\boldsymbol{h}}{6}(\boldsymbol{k_1} + 2\boldsymbol{k_2} + 2\boldsymbol{k_3} + \boldsymbol{k_4}) \quad (17)$$

NUMERICAL RESULTS
Each of these methods has unique advantages and is suitable for various problems in the field of numerical analysis. The choice of the appropriate method depends on the specific characteristics of the differential equation to be solved, as well as the required level of accuracy. We will present some examples, where MATLAB will be used to compute all the results.

Example1: Consider the Bernoulli differential equation

$$\frac{dy}{dx} = y + xy^{1/2} \quad (18)$$

with initial condition $y(0) = 1$. Its general solution is given by

$$y(x) = (e^{\frac{x}{2}} - x - 2)^2 \quad (19)$$

Table 1. The scheduling error rates include techniques such as the Heun method, the Midpoint method, the Ralston method, and the fourth-order Runge-Kutta method, in addition to the exact solution of the problem in question on $h = 0.1$.

h	x	Exact	Abs. Error Heun	Abs. Error Midpoint	Abs. Error Ralston's	Abs. Error RK th4
0.1	0.1	1.099832	9.466653e-03	9.357028e-03	9.412159e-03	9.719653e-03
0.1	0.2	1.198650	3.759466e-02	3.737013e-02	3.748306e-02	3.814090e-02
0.1	0.3	1.295421	8.401656e-02	8.366973e-02	8.384420e-02	8.491040e-02
0.1	0.4	1.389091	1.492399e-01	1.487611e-01	1.490020e-01	1.505516e-01
0.1	0.5	1.478594	2.345648e-01	2.339418e-01	2.342552e-01	2.363828e-01
0.1	0.6	1.562853	3.420738e-01	3.411780e-01	3.416852e-01	3.445085e-01
0.1	0.7	1.640787	4.746852e-01	4.737256e-01	4.742083e-01	4.778731e-01
0.1	0.8	1.711322	6.362638e-01	6.351046e-01	6.356877e-01	6.403735e-01
0.1	0.9	1.773392	8.317983e-01	8.304127e-01	8.311096e-01	8.370378e-01

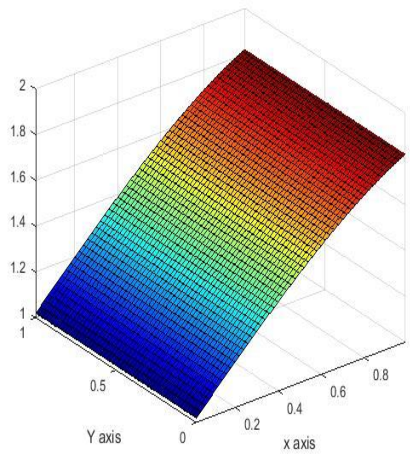

Figure. 1: Approximate Solution

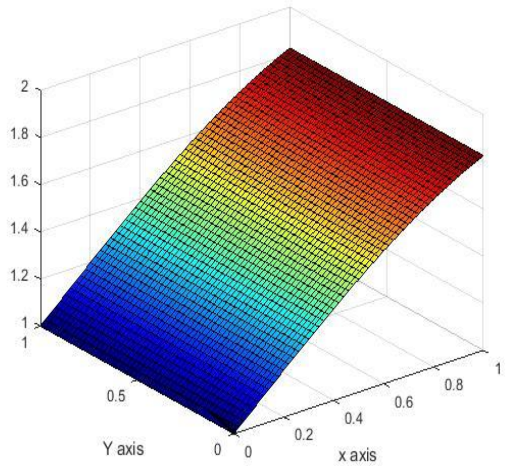

Figure. 2: Exact Solution

Example 2: Consider the Bernoulli differential equation

$$\frac{dy}{dx} = x^3y^3 - xy \quad (20)$$

with initial condition $y(0) = 1$. Its general solution is given by

$$y(x) = \frac{1}{(1+x^2)} \quad (21)$$

Table 2. Scheduling of Heun's methods, the midpoint method, Ralston's methods, and fourth-order Runge-Kutta methods, in addition to the exact solution for the second example on $h = 0.1$

h	x	Exact	Heun	Midpoint	Ralston's	RK th4
0.1	0.1	9.900990 e-01	9.950500e-01	9.950125 e-01	9.950281e-01	9.950371 e-01
0.1	0.2	9.615384 e-01	9.806545e-01	9.804887 e-01	9.805659e-01	9.805805 e-01
0.1	0.3	9.174311 e-01	9.579993e-01	9.576429 e-01	9.578139e-01	9.578261 e-01
0.1	0.4	8.620689 e-01	9.287712e-01	9.281988 e-01	9.284773e-01	9.284764 e-01
0.1	0.5	8.000000 e-01	8.948478e-01	8.940668 e-01	8.944503e-01	8.944268 e-01
0.1	0.6	7.352941 e-01	8.580302e-01	8.570678 e-01	8.575432e-01	8.574925 e-01
0.1	0.7	6.711409 e-01	8.198675e-01	8.187579 e-01	8.193084e-01	8.192314 e-01
0.1	0.8	6.097560 e-01	7.815814e-01	7.803550 e-01	7.809656e-01	7.808683 e-01
0.1	0.9	5.524861 e-01	7.440642e-01	7.427418 e-01	7.434020e-01	7.432936 e-01

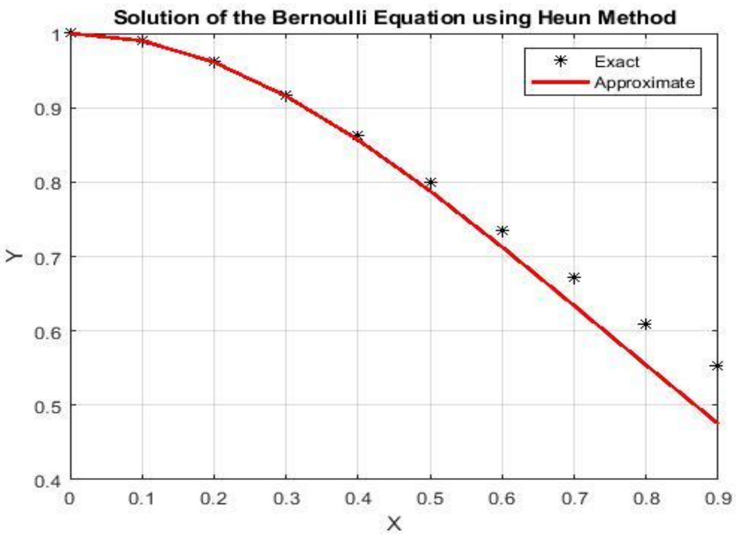

Figure 3. Compare Approximate Solutions and Exact Solution for Example 2

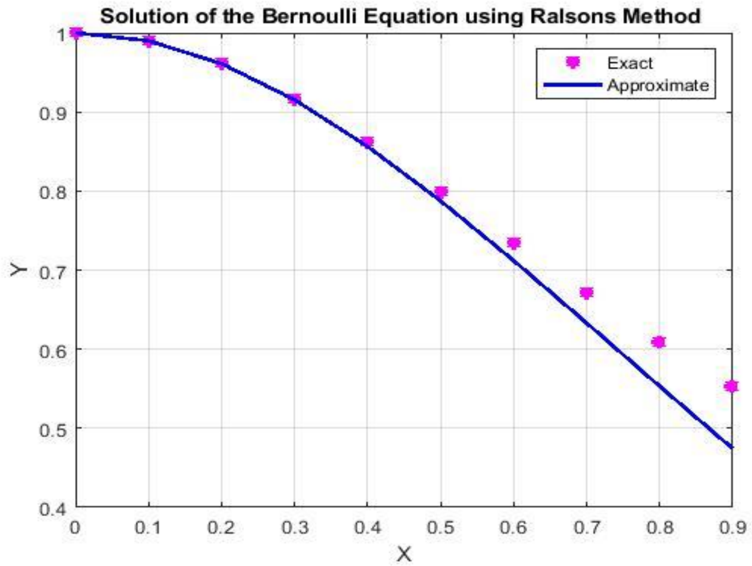

Figure. 4: Compare Approximate Solutions and Exact Solution for Example 2

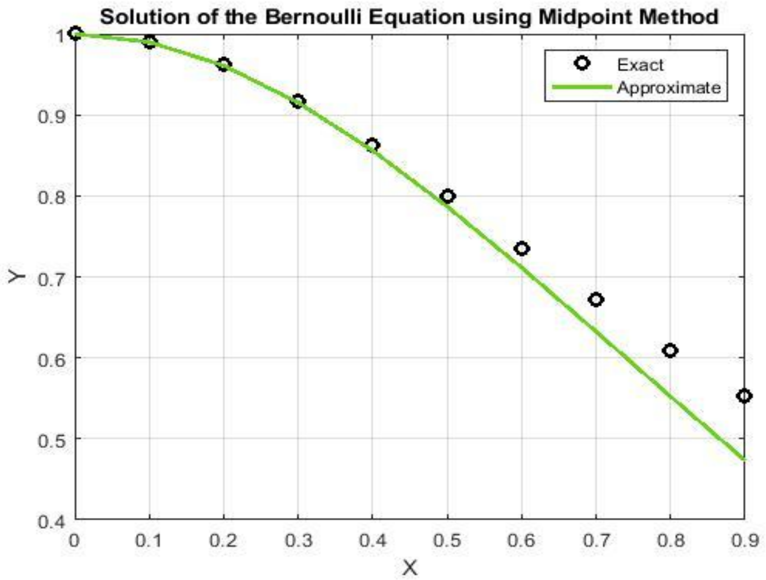

Figure. 5: Compare Approximate Solutions and Exact Solution for Example 2

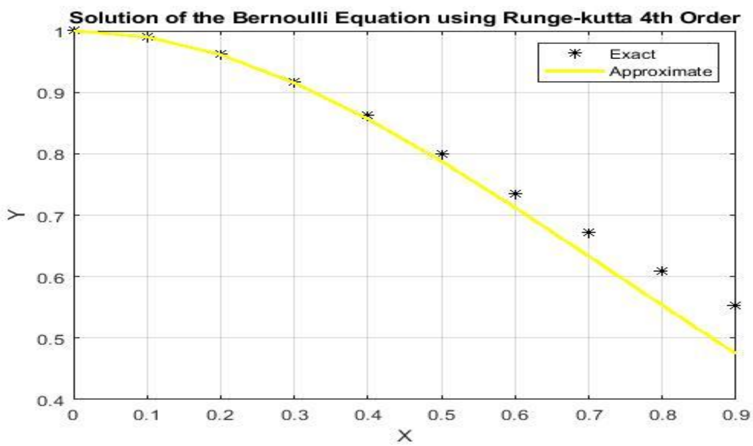

Figure. 6: Compare Approximate Solutions and Exact Solution for Example 2

Example 3: Consider the Bernoulli differential equation

$$\frac{dy}{dx} = 2xy + 2x^3y^2 \qquad (22)$$

with initial condition $y(0) = 1$. Its general solution is given by

$$y(x) = \frac{1}{(1-x^2)} \qquad (23)$$

Table 3: The scheduling of Heun, Midpoint, Ralston's, fourth-order RK methods, and Exact Solution for Example 3 on $h = 0.1$.

h	x	Exact	Heun	Midpoint	Ralston's	RK th4
0.1	0.1	1.010101	1.010100	1.010025	1.010056	1.010101
0.1	0.2	1.041666	1.041762	1.041334	1.041534	1.041666
0.1	0.3	1.098901	1.099200	1.098017	1.098582	1.098901
0.1	0.4	1.190476	1.191092	1.188443	1.189717	1.190477
0.1	0.5	1.333333	1.334315	1.328745	1.331430	1.333335
0.1	0.6	1.562500	1.563418	1.551479	1.557231	1.562505
0.1	0.7	1.960784	1.958217	1.930123	1.943609	1.960790
0.1	0.8	2.777777	2.745283	2.664861	2.703140	2.777585
0.1	0.9	5.263157	4.847882	4.506619	4.665764	5.248673

Table 4: Tabulation Heun, Midpoint, Ralston's, and RK th4 error methods with the Exact solution for Example 3 on $h = 0.1$

H	x	Exact	Abs. Error Heun	Abs. Error Midpoint	Abs. Error Ralston's	Abs. Error RK th4
0.1	0.1	1.010101	1.000000e-06	7.525000e-05	4.431249e-05	4.122993e-08
0.1	0.2	1.041666	9.219737e-05	3.189751e-04	1.270442e-04	1.707306e-07
0.1	0.3	1.098901	2.726972e-04	8.043720e-04	2.903775e-04	4.214655e-07
0.1	0.4	1.190476	5.174764e-04	1.707634e-03	6.375365e-04	8.823297e-07
0.1	0.5	1.333333	7.364984e-04	3.441058e-03	1.427066e-03	1.762172e-06
0.1	0.6	1.562500	5.881337e-04	7.053272e-03	3.371750e-03	3.352860e-06
0.1	0.7	1.960784	1.308959e-03	1.563680e-02	8.758906e-03	2.972069e-06
0.1	0.8	2.777777	1.169794e-02	4.065000e-02	2.686932e-02	6.925501e-05
0.1	0.9	5.263157	7.890228e-02	1.437423e-01	1.135046e-01	2.752018e-03

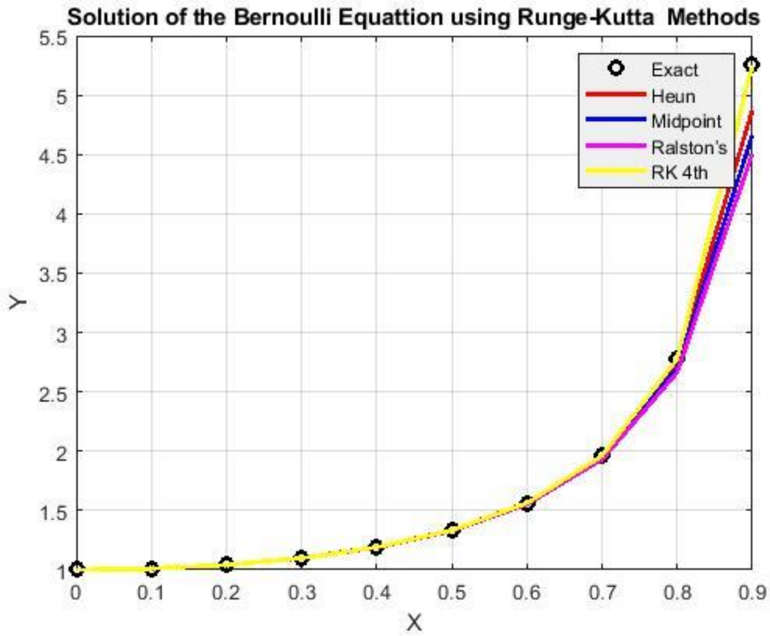

Figure. 7: Compare Approximate Solutions and Exact Solution for Example 3

CONCLUSION

This study highlights the effectiveness of various numerical methods in solving nonlinear equations. By understanding the strengths and weaknesses of each method, the study compares the efficiency and accuracy of these approaches through numerical experiments on a variety of nonlinear differential equations. The results indicate that the RK4 method generally provides the highest level of accuracy, while the Heun, midpoint, and Ralston methods are useful in specific applications where computational resources are limited. Practitioners can choose the most suitable method for their specific problem, balancing accuracy and computational efficiency. Future studies could explore hybrid methods that combine the advantages of these techniques to enhance performance.

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مقارنة بين طرق رونج-كوتا لحل المعادلات غير الخطية

صفاء أبوعمره[1]*، تبرة المجبري[2]، منال علي[1]

[1]قسم الرياضيات، كلية العلوم، جامعة عمر المختار، البيضاء، ليبيا
[2]قسم الهندسة وعلم المواد، تخصص الرياضيات، كلية الهندسة، جامعة عمر المختار، البيضاء، ليبيا

المستخلص
يتناول هذا البحث الحلول العددية للمعادلات التفاضلية العادية غير الخطية باستخدام أربع طرق مميزة: طريقة هيون، وطريقة نقطة المنتصف، وطريقة رالستون، وطريقة رونج-كوتا من الدرجة الرابعة. ونظرًا لتعقيد وانتشار المعادلات غير الخطية في مختلف المجالات العلمية، فإن التقنيات العددية الفعالة ضرورية للحصول على حلول دقيقة. يساهم هذا البحث في فهم الأساليب العددية في حل المعادلات التفاضلية المعقدة، ومساعدة الممارسين في اختيار النهج الأكثر ملاءمة لاحتياجاتهم.
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