Review study

The Energy Principle of MHD Instabilities

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ABSTRACT

In this work the MHD instability problem is reviewed, given some static equilibrium parameters (ρ_0 , p_0 , $\vec{B}_0$ and $v_0 = 0$), we study this equilibrium for small perturbations to see if these perturbations grow or decay. Among the several approaches, the energy principle is used, and the criteria for its application are recovered. This condition is applied in the study of the interchange, sausage and the kink instabilities.

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INTRODUCTION

We take more general review of the MHD instability problem: given some static equilibrium (ρ_0 , p_0 , $\vec{B}_0$ and $v_0 = 0$), we will study this equilibrium for small perturbations to see if these perturbations grow or decay. There are two ways to do it: Firstly, try out the normal-mode analysis, i.e., linearize the MHD equations around the given equilibrium, and see if any of the frequencies turn out to be complex, with positive imaginary parts (growth rates) [9]. This approach has the advantage of being direct and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do: just being able to prove that some configuration is stable or that certain types of perturbations might grow [2,5]. Hence the second approach: Secondly check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system increasing. If so, then the equilibrium is stable—this is called the energy principle and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy decreasing, that equilibrium is unstable. The advantage of this second approach is that we do not need to solve the linearized MHD equations in order to find instability [6, 8], just examine the properties of the perturbed energy functional. It should be already quite clear how to do the normal-mode analysis, at least in principle, so the second approach is used.

Energy Principle

Consider what the total energy in MHD is [3]:

$$\mathcal{E} = \int dr^3 \left(\frac{1}{2}\rho v^2 + \frac{1}{8\pi}B^2 + \frac{1}{\gamma-1}p\right) = \int dr^3 \left(\frac{1}{2}\rho v^2 + \psi\right) \qquad (1)$$

As usual, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements ξ, $\vec{v} = \frac{\partial\vec{\xi}}{\partial t}$, and that by definition of ξ we get:

$$\varepsilon = \int dr^3 \frac{1}{2}\rho_0 \left|\frac{\partial\vec{\xi}}{\partial t}\right|^2 + \psi_0 + \hat{\psi}_1[\vec{\xi}] + \hat{\psi}_2[\vec{\xi},\vec{\xi}] + \cdots \qquad (2)$$

Where we have kept terms up to second order in ξ and so ψ_0 is the equilibrium part of ψ (i.e., its value for $\vec{\xi} = 0$), $\hat{\psi}_1[\xi]$ is linear in $\vec{\xi}$ and $\hat{\psi}_2[\vec{\xi},\vec{\xi}]$ is quadratic term, etc. Energy must be conserved for all orders, so:

$$\frac{d\varepsilon}{dt} = \int dr^3 \ \vec{\mathcal{F}}[\xi]\cdot\frac{\partial\vec{\xi}}{\partial t} + \hat{\psi}_1\left[\frac{\partial\vec{\xi}}{\partial t}\right] + \hat{\psi}_2\left[\frac{\partial\vec{\xi}}{\partial t},\vec{\xi}\right] + \hat{\psi}_2\left[\vec{\xi},\frac{\partial\vec{\xi}}{\partial t}\right] + \cdots = 0 \qquad (3)$$

$Where,\ \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} \equiv \vec{\mathcal{F}}[\xi].$

This must be true at all times, including at t = 0, when $\vec{\xi}$ and $\partial\vec{\xi}/\partial t$ can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions $\vec{\xi}$ and $\vec{\chi}$, [9]:

$$\int dr^3\, \vec{\chi}\cdot\vec{\mathcal{F}}[\vec{\xi}] + \hat{\psi}_1[\vec{\chi}] + \hat{\psi}_2[\vec{\chi},\vec{\xi}] + \hat{\psi}_2[\vec{\xi},\vec{\chi}] + \cdots = 0 \qquad (4)$$

In the first order, this tells us that

$$\hat{\psi}_1[\vec{\chi}] = 0 \qquad (5)$$

This means that there are no first-order energy perturbations. In the second order, we have

$$\int dr^3\, \vec{\chi}\cdot\vec{\mathcal{F}}[\vec{\xi}] = -\hat{\psi}_2[\vec{\chi},\vec{\xi}] - \hat{\psi}_2[\vec{\xi},\vec{\chi}] \qquad (6)$$

Let $\vec{\chi} = \vec{\xi}$ implies

$$\hat{\psi}_2[\vec{\xi},\vec{\xi}] = -\frac{1}{2}\int dr^3\, \vec{\xi}\cdot\vec{\mathcal{F}}[\vec{\xi}] \qquad (7)$$

This is the part of the perturbed energy in Eq. (2) that can be both positive and negative.
The energy principle implies that if:

$$\hat{\psi}_2[\vec{\xi},\vec{\xi}] > 0, \text{ for any } \vec{\xi} \qquad (8)$$

The result is a stable equilibrium.

(a) Instability. (b) "Over stability"

Figure 1. MHD instabilities

Properties of the Force Operator

Since the right-hand side of Eq.(6) is symmetric with respect to interchanging $\vec{\xi} \leftrightarrow \vec{\chi}$ so must be the left-hand side:

$$\int dr^3\, \vec{\chi}\cdot\vec{\mathcal{F}}[\vec{\xi}] = \int dr^3\, \vec{\xi}\cdot\vec{\mathcal{F}}[\vec{\chi}] \qquad (9)$$

Therefore, the force operator $\vec{\mathcal{F}}[\xi]$ is self-adjoint. By definition:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} \equiv \vec{\mathcal{F}}[\vec{\xi}] \qquad (10)$$

The eigenmodes of this operator satisfy

$$\vec{\xi}(r,t) = \vec{\xi}_n exp^{-i\omega_n t} \implies \vec{\mathcal{F}}[\vec{\xi}_n] = \rho_0 \omega_n^2 \vec{\xi}_n \qquad (11)$$

As always for self-adjoint operators, their eigenvalues $\{\omega_n^2\}$ are real, and the eigenmodes $\{\vec{\xi}_n\}$ are orthogonal. This result implies that, if any MHD equilibrium is unstable, at least one of the eigenvalues must be $\omega_n^2 < 0$ and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 1a), rather than growing oscillations "over stabilities"; see (Fig. 1b).

Calculation of $\hat{\psi}_2[\vec{\xi},\vec{\chi}]$

Now that we know that we need the sign of $\hat{\psi}_2\xi[\vec{\xi},\vec{\chi}]$ to ascertain stability (or otherwise), it is worth working out $\hat{\psi}_2[\vec{\xi},\vec{\chi}]$ as an explicit function of $\vec{\xi}$. It is a second-order quantity, but Eq. (7) tells us that all we need to calculate is $\vec{\mathcal{F}}[\vec{\xi}]$ to first order in $\vec{\xi}$, i.e., we just need to linearize the MHD equations around an arbitrary static equilibrium.

Linearized MHD Equations

Assuming that equilibrium quantities $\vec{v}_0 = 0$ and $(\rho_0, p_0, \vec{B}_0)$ are constant in time and using the general perturbations function as for a given mode (keeping only first order terms);

$f(\vec{\xi}, t) = f_0 + \hat{f}\, exp^{i(\vec{k}\cdot\vec{\xi} - \omega t)}$. Hence, using $\vec{v} = \frac{\partial \vec{\xi}}{\partial t}$ we obtain;

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \Rightarrow \quad \frac{\partial \hat{\rho}}{\partial t} = -\vec{\nabla}\cdot\rho_0 \frac{\partial \vec{\xi}}{\partial t} \quad \Rightarrow \hat{\rho} = -\vec{\nabla}\cdot(\rho_0\vec{\xi}) \tag{20}$$

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)p = -\gamma p \vec{\nabla}\cdot\vec{v} \quad \Rightarrow \quad \hat{p} = -\vec{\xi}\cdot\vec{\nabla}p_0 - \gamma p_0 \vec{\nabla}\cdot\vec{\xi} \tag{21}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) \quad \Rightarrow \quad \hat{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0) \tag{22}$$

As the second part of Eq.(1) suggest, $\hat{\psi} = \int dr^3 \left(\frac{1}{8\pi}(\widehat{B^2}) + \frac{1}{\gamma-1}\hat{p}\right)$ must be some operator involving $\vec{\xi}$ and its gradients.

Remember $\hat{\rho}$, $\hat{p}$ and $\hat{B}$ are all vectors expressed as linear operators in $\vec{\xi}$.

Finally, add gravity to the force term in the momentum equation and linearizing:

$$\rho\left(\frac{\partial v}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = \vec{\nabla}p + \frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B} + \rho\vec{g} \tag{23}$$

This leads to:

$$\rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2} \equiv \vec{\mathcal{F}}[\vec{\xi}] = -\vec{\nabla}\hat{p} + \frac{1}{4\pi}(\vec{\nabla}\times\vec{B}_0)\times\hat{B} + \frac{1}{4\pi}(\vec{\nabla}\times\hat{B})\times\vec{B}_0 + \hat{\rho}\vec{g}$$
$$= \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p_0 + \gamma p_0\vec{\nabla}\cdot\vec{\xi}) - \vec{g}\vec{\nabla}\cdot(\rho_0\vec{\xi}) + \frac{1}{c}\vec{j}_0\times\hat{B} + \frac{1}{4\pi}(\vec{\nabla}\times\hat{B})\times\vec{B}_0 \tag{24}$$

Where $\vec{j}_0 = \frac{c}{4\pi}(\vec{\nabla}\times\vec{B}_0)$, we have used Eq.(20) and Eq.(21) for $\hat{\rho}$ and $\hat{p}$ respectively, and $\hat{B}$ is given as:

$$\hat{B} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}_0).$$

Energy Perturbation

Substituting Eq.(24) into Eq.(7) use integration by parts and some vector identities manipulation to arrive at the energy perturbation expression,[7]:

$$\hat{\psi}_2[\vec{\xi},\vec{\chi}] = \frac{1}{2}\int dr^3 \begin{Bmatrix} (\vec{\xi}\cdot\vec{\nabla}p_0)\vec{\nabla}\cdot\vec{\xi} + \gamma p_0(\vec{\nabla}\cdot\vec{\xi})^2 \\ +(\vec{g}\cdot\vec{\xi})\vec{\nabla}\cdot(\rho_0\vec{\xi}) + \frac{1}{c}\vec{j}_0\cdot(\vec{\xi}\times\hat{B}) + \frac{|\hat{B}|^2}{4\pi} \end{Bmatrix} \tag{26}$$

Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilizing. The terms that are not sign definite and so potentially destabilizing involve equilibrium gradients of pressure, density and magnetic field (currents), [9].

What we need to show now is to calculate $\hat{\psi}_2[\vec{\xi},\vec{\chi}]$ according to Eq.(26) for any equilibrium that of interest and see if it can be negative for any class of perturbations (or positive for all perturbations). The first statement will lead to instability i.e. $\hat{\psi}_2[\vec{\xi},\vec{\chi}] < 0$ and the second will lead to stability, i.e. $\hat{\psi}_2[\vec{\xi},\vec{\chi}] > 0$.

Interchange Instabilities

To put the energy principle to work to classify stability, we will start to consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic stratified atmosphere, [5]:

$\rho_0 = \rho_0(z)$ and $p_0 = p_0(z)$ satisfying

$$\frac{dp_0}{dz} = -\rho_0\vec{g} \quad , (\vec{g} = -g\hat{k}) \tag{27}$$

With $\vec{B}_0 = 0$ and the hydrostatic equilibrium Eq. (27) and Eq.(26) becomes

$$\hat{\psi}_2 = \frac{1}{2}\int dr^3 \left\{2p_0'\,\xi_z\vec{\nabla}\cdot\vec{\xi} + \gamma p_0(\vec{\nabla}\cdot\vec{\xi})^2 - \rho_0' g\xi_z^2\right\} \tag{28}$$

Where, we have used $\rho_0 g = -p_0'$. We see that $\hat{\psi}_2$ depends on ξ_z and $\vec{\nabla}\cdot\vec{\xi}$. Let us treat them as independent variables and minimise $\hat{\psi}_2$ with respect to them (i.e., seek the most unstable possible situation):

$$\frac{\partial\hat{\psi}_2}{\partial(\vec{\nabla}\cdot\vec{\xi})} = 2p_0'\,\xi_z + 2\gamma p_0(\vec{\nabla}\cdot\vec{\xi}) = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{\xi} = \frac{p_0'}{\gamma p_0}\xi_z \tag{29}$$

Substituting this back into Eq.(28), we get

$$\hat{\psi}_2 = \frac{1}{2}\int dr^3\left(-\frac{p_0'^2}{\gamma p_0} - \rho_0' g\xi_z^2\right) = \frac{1}{2}\int dr^3 \frac{\rho_0 g}{\gamma}\frac{d}{dz}\left(ln\frac{p_0}{\rho_0^\gamma}\right)\xi_z^2 = \frac{1}{2}\int dr^3\frac{\rho_0 g}{\gamma}\frac{d}{dz}(lns_0)\xi_z^2 \quad (30)$$

Where, $s_0 \equiv \frac{p_0}{\rho_0^\gamma}$ give the entropy function. By the Energy Principle, the system is stable if

$$\hat{\psi}_2 > 0 \implies \frac{d}{dz}(lns_0) > 0 \quad (31)$$

The inequality Eq.(31) is the Schwarzschild criterion for convective stability. If this criterion is broken, there will be instability, called the interchange instability. When the Schwarzschild criterion is broken $(\hat{\psi}_2 < 0)$, the physical situation is as the following. Recalling Eq.(20) and Eq.(21) and take the displacements which minimize $\hat{\psi}_2$ given by Eq.(29) to get:

$$\frac{\hat{p}}{p_0} = -p_0' p_0 \xi_x - \gamma\vec{\nabla}\cdot\vec{\xi} = 0 \quad (32)$$

$$\frac{\hat{\rho}}{\rho_0} = -\rho_0'\rho_0\xi_x - \gamma\vec{\nabla}\cdot\vec{\xi} = \frac{1}{\gamma}\left(-\frac{1}{\gamma}\frac{\rho_0'}{\rho_0} + \frac{p_0'}{p_0}\right) = \frac{1}{\gamma}\frac{d(lns_0)}{dz}\xi_z \quad (33)$$

The found perturbations are not sound waves since there is no pressure change. They are local increase or decrease in density for blobs (column) of fluid that fall ($\xi_z < 0$) or rise ($\xi_z > 0$), respectively. That is, if we imagine a blob of fluid slowly rising (adiabatically slowly, so $\hat{p} = 0$) from the denser regions of the atmosphere to the less dense upper ones, then to stay in pressure balance with its surroundings will require the blob to expand ($\hat{\rho} < 0$) or contract ($\hat{\rho} > 0$). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly) and the system will be unstable $\left(\frac{d(lns_0)}{dz} < 0\right)$, see Fig.2. The direction of the entropy gradient determines which of these two scenarios is realized,[2].

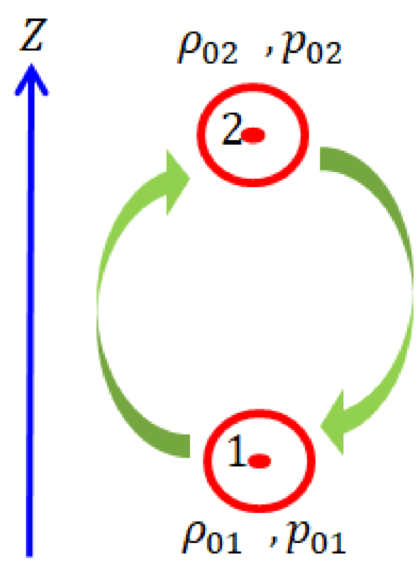

Figure 2. Interchange instability

Instabilities of a Pinch

Consider the stability of the z-pinch equilibrium, review by Haines, [4]:

$$\vec{B}_0 = B_0(r)\hat{e}_\theta\,,\ \vec{j}_0 = j_0(r)\hat{e}_z = \frac{c}{4\pi}\frac{(r\vec{B}_0)'}{r}\hat{e}_z \ \text{ and } \ p_0'(r) = -\frac{1}{c}j_0 B_0 = -\frac{B_0(r\vec{B}_0)'}{4\pi r} \quad (34)$$

Working in cylindrical coordinates, we must first write all the terms in Eq.(26) in these coordinates using the equilibrium terms in Eq.(34):

$$(\vec{\xi}\cdot\vec{\nabla}p_0)\vec{\nabla}\cdot\vec{\xi} = p_0'\frac{\xi_r^2}{r} + p_0'\xi_r\left(\frac{\partial\xi_r}{\partial r} + \frac{\partial\xi_z}{\partial z}\right) \quad (35)$$

$$\gamma p_0\left(\vec{\nabla}\cdot\vec{\xi}\right)^2 = \gamma p_0\left(\frac{1}{r}\frac{\partial(r\xi_r)}{\partial r} + \frac{1}{r}\frac{\partial(\xi_\theta)}{\partial\theta} + \frac{\partial\xi_z}{\partial z}\right)^2 \quad (36)$$

$$\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0) = \hat{e}_r\left(\frac{1}{r}\frac{\partial(\xi_r B_0)}{\partial\theta}\right) - \hat{e}_\theta\left(\frac{\partial(\xi_z B_0)}{\partial z} + \frac{\partial(\xi_r B_0)}{\partial r}\right) + \hat{e}_z\left(\frac{1}{r}\frac{\partial(\xi_z B_0)}{\partial\theta}\right) \quad (37)$$

$$\frac{1}{c}\vec{j}_0 \cdot (\vec{\xi} \times \hat{B}) = p_0' \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r}\right) + \frac{p_0' B_0'}{B_0} \xi_r^2 \quad (38)$$

$$\frac{|\hat{B}|^2}{4\pi} = \frac{B_0^2}{4\pi r^2}\left\{\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right\} + \frac{B_0^2}{4\pi}\left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B_0'}{B_0}\right)^2 \quad (39)$$

Combining all this together, to obtain:

$$\hat{\psi}_2 = \frac{1}{2}\int dr^3 \left\{ \begin{array}{l} 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi}\left(r \frac{\partial}{\partial r}\frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right)^2 \\ +\gamma p_0 \left(\frac{1}{r}\frac{\partial (r\xi_r)}{\partial r} + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z}\right)^2 + \frac{B_0^2}{4\pi r^2}\left(\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right) \end{array} \right\} \quad (40)$$

This is the energy perturbation expression, obtained in cylindrical coordinates [7].

Sausage Instability

Let us first consider axisymmetric perturbations: $\frac{\partial}{\partial \theta} = 0$. Then $\hat{\psi}_2$ depends on two variables only: ξ_r and

$$\chi = \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z} \quad (41)$$

Indeed, unpacking all the r derivatives in Eq.(40), we get

$$\hat{\psi}_2 = \frac{1}{2}\int dr^3 \left\{2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi}\left(\chi - \frac{\xi_r}{r}\right)^2 + \gamma p_0 \left(\chi + \frac{\xi_r}{r}\right)^2\right\} \quad (42)$$

We shall treat ξ_r and χ as independent variables and minimise $\hat{\psi}_2$ with respect to χ:

$$\frac{\partial \hat{\psi}_2}{\partial \chi} = 2\frac{B_0^2}{4\pi}\left(\chi - \frac{\xi_r}{r}\right) + 2\gamma p_0 \left(\chi + \frac{\xi_r}{r}\right) = 0 \implies \chi = \frac{1-\gamma\beta/2}{1+\gamma\beta/2}\frac{\xi_r}{r} \quad (43)$$

Where, $\beta = \frac{8\pi p_0}{B_0^2}$. Substituting the obtained value of χ into Eq.(42), we have:

$$\hat{\psi}_2 = \frac{1}{2}\int dr^3\, p_0 \left(r\frac{dp_0}{dr} + \frac{2\gamma}{1+\gamma\beta/2}\right)\frac{\xi_r^2}{r^2} \quad (44)$$

There will be instability ($\hat{\psi}_2 < 0$) if:

$$-r\frac{dp_0}{dr} > \frac{2\gamma}{1+\gamma\beta/2} \quad (45)$$

i.e., when the pressure gradient is too steep, the equilibrium is unstable. Recall that the perturbations that we have identified as making $\hat{\psi}_2 < 0$ are axisymmetric, have some radial and axial displacements and are compressible: from Eq. (43),

$$\vec{\nabla} \cdot \vec{\xi} = \chi + \frac{\xi_r}{r} = \frac{2}{1+\gamma\beta/2}\frac{\xi_r}{r} \quad (46)$$

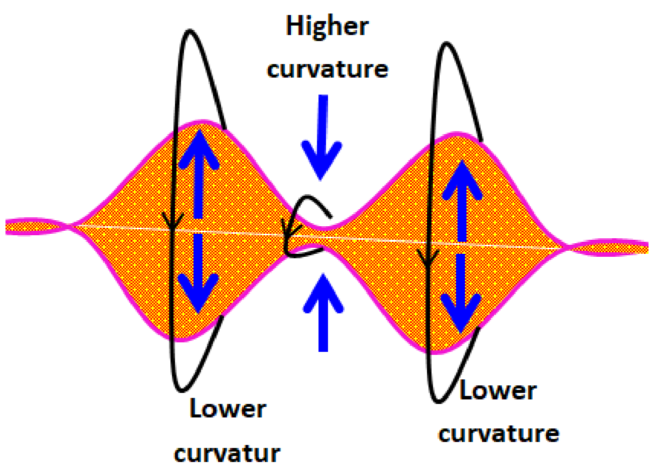

Figure 3. Sausage instability

They are illustrated in Fig.3. The mechanism of this aptly named sausage instability is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further [1].

Kink Instability

Now consider non-axisymmetric perturbations ($\frac{\partial}{\partial\theta} \neq 0$) to see what other instabilities might exist. First of all, since we now have θ variation, $\hat{\psi}_2$ depends on ξ_θ. However, in Eq.(40), ξ_θ only appears in the third term, where it is part of $\vec{\nabla}\cdot\vec{\xi}$, which enters quadratically and with a positive coefficient γp_0 We can treat $\vec{\nabla}\cdot\vec{\xi}$ as an independent variable, alongside ξ_r and ξ_z, and minimise $\hat{\psi}_2$ with respect to it. Obviously, the energy perturbation is minimal when

$$\vec{\nabla}\cdot\vec{\xi} = 0 \quad (47)$$

i.e., the most dangerous non-axisymmetric perturbations are incompressible. To carry out further minimization of $\hat{\psi}_2$, it is convenient to Fourier transform our displacements in the θ and z directions—both are directions of symmetry (i.e., the equilibrium profiles do not vary in these directions), thus:

$$\vec{\xi} = \sum_{m,k}\vec{\xi}_{mk}(r)\, exp^{i(m\theta+kz)} \quad (48)$$

Then Eq.(40) (with $\vec{\nabla}\cdot\vec{\xi} = 0$) becomes, by Parseval's theorem (the operator $\vec{\mathcal{F}}[\xi]$ being self-adjoint:

$$\hat{\psi}_2 = \frac{1}{2}\sum_{m,k} 2\pi l_z \int_0^\infty r dr \left\{2p_0' \frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi}\left(\left|r\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right) + ik\xi_z\right|^2 + \frac{m^2}{r^2}\left(|\xi_r|^2 + |\xi_z|^2\right)\right)\right\} \quad (49)$$

As ξ_z and ξ_z^* only appear algebraically in Eq. (49) (no r derivatives), it is easy to minimize $\hat{\psi}_2$ with respect to them: setting the derivative of the integrand with respect to either ξ_z or ξ_z^* to zero, we obtain:

$$ik\left(r\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right) + ik\xi_z\right) + \frac{m^2}{r^2}\xi_z = 0 \implies \xi_z = \frac{ikr^3}{m^2+k^2r^2}\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right) \quad (50)$$

Putting this back into Eq. (49) and assembling terms, we have:

$$\hat{\psi}_2 = \sum_{m,k}\pi l_z \int_0^\infty r dr \left\{2p_0\left(\frac{rp_0'}{p_0} + \frac{m^2}{\beta}\right)\frac{|\xi_r|^2}{r^2} + \frac{B_0^2}{4\pi}\left(\frac{m^2}{m^2+k^2r^2}\right)\left|r\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right)\right|^2\right\} \quad (51)$$

The second term here is always stabilizing. The most unstable modes will be ones with $k \to \infty$ for which the stabilizing term is as small as possible. The remaining term will allow $\hat{\psi}_2 < 0$ and, therefore, there is instability, if:

$$-r\frac{d}{dr}(ln p_0) > \frac{m^2}{\beta} \quad (52)$$

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest m with, $m = 1$.

The unstable perturbations are incompressible,[6]:

$$\vec{\nabla}\cdot\vec{\xi} = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{im}{r}\xi_\theta + ik\xi_z = 0 \quad (53)$$

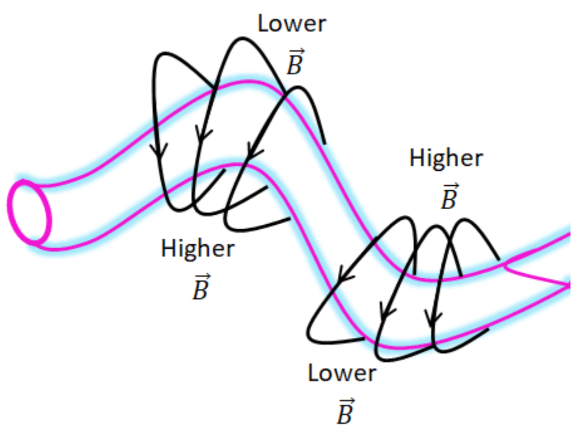

Figure 4. Kink instability

Setting, $m = 1$ and using Eq.(50), we get:

$$(i\xi_\theta)_{k\to\infty} \approx -\frac{\partial}{\partial r}(r\xi_r) + r^2\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right) \approx 2\xi_r \quad \text{and} \quad \xi_z \ll \xi_r \quad (54)$$

The instability action is as follows: the flux surfaces are bent, with a twist (to remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher $\vec{B}$ to those with lower $\vec{B}$, thermal pressure in the latter (convex) areas becomes uncompensated; the field lines open up further, etc. This is called the kink instability, [10]; see Fig.4.

CONCLUSION

Among the several approaches, the energy principle is used. The energy principle is applied in the study of the interchange, sausage and the kink instabilities. We study this equilibrium for small perturbations to see if these perturbations grow or decay.

Disclaimer

The article has not been previously presented or published and is not part of a thesis project.

Disclosure statement

None to declare

Competing interest

There are no financial, personal, or professional conflicts of interest to declare.

REFERENCES

1. Boldyrev S, Loureiro N. Calculations in the theory of tearing instability. J Phys Conf Ser. 2018;1100 012003.
2. Chandrasekhar S. Hydrodynamic and Hydro magnetic Stability. Dover. 2003. ISBN; 10-048664071X
3. Davidson P. Introduction to Magneto hydrodynamics (2nd edition). (Cambridge: Cambridge University Press., 2016)
4. Haines M. A review of the dense Z-pinch. Plasma Phys. Control. Fusion 2011;53;093001.
5. Hosking D, Schekochihin A. Reconnection-controlled decay of magneto hydrodynamic turbulence and the role of invariants. Phys. Rev. 2021; X11,041005.
6. Kunz M. Schekochihin A, Chen C, Abel I, Cowley S. Inertial-range kinetic turbulence in pressure-anisotropic astrophysical plasmas. J. Plasma Phys. 2015;81;325810501.
7. Kulsrud R. Plasma Physics for Astrophysics. Princeton: Princeton University Press (2005).
8. Loureiro N, Schekochihin S, Cowley S. Instability of current sheets and formation of plasmoid chains. Phys Plasmas. 2007;14; 100703.
9. Parra F, Collisional Plasma Physics. Lecture Notes for an Oxford M Math Phys course (2019.
10. Uzdensky D, Loureiro N. Magnetic reconnection onset via disruption of a forming current sheet by the tearing instability. Phys Rev Lett. 2016;116;105003.