Alqalam Journal of Medical and Applied Sciences. 2025;8(3):1785-1794
https://doi.org/10.54361/ajmas.258374

Original article

Analytical Approximate Solution of Conformable Fractional Fifth-Order Korteweg-de Vries Equation via the ARA-residual Power Series Method

Albatool Alfartas*, **Asma Agsaisib**, **Yasmina Bader**

Department of Mathematics, University of Derna, Derna, Libya
Corresponding Email. a.alfartas@uod.edu.ly

Abstract
In this paper, we introduce an analytical method for deriving an approximate solution to the time-dependent fifth-order Korteweg-de Vries (fKdV) equation using the conformable fractional derivative (CFD) via the ARA-residual power series method (ARA-RPSM). The proposed method operates by initially applying the ARA-transform to the given fKdV equation. Subsequently, approximate series solutions are derived using Taylor's expansion. These series solutions are then converted back into the original domain through the inverse ARA-transform. It is a general method for time-dependent nonlinear differential equations and has wide applicability. The efficiency and flexibility of this method make it useful for a wide range of time-dependent nonlinear differential equations. To demonstrate its effectiveness, we apply it to the time-dependent fKdV equation, showcasing how it generates reliable and accurate series solutions quickly.
Keywords. Fifth-Order Korteweg-De Vries Equation, Conformable Fractional Derivative, Approximate Solutions; ARA-Residual Power Series; Fractional Calculus.

Introduction

Fractional partial differential equations (FPDEs) are like regular PDEs but use fractional (non-integer) derivatives. This makes them well-suited for systems with memory and long-range effects—such as anomalous diffusion—where the present depends on the past and interactions are not purely local. Because of this, FPDEs help explain behaviors across scales and are widely used in physics, finance, biology, and engineering. The mathematical foundation is fractional calculus, which extends differentiation and integration beyond whole numbers [1–3], and many real systems show history-dependent behavior that motivates these models [4–12].

Fractional differential equations have been studied using several definitions of fractional differentiation, including the Riemann–Liouville [13] and Caputo [14,15] operators. Among recent proposals, the conformable derivative [16] has attracted particular attention. Broadly, fractional derivatives can be classified as nonlocal or local: the Riemann–Liouville and Caputo operators are nonlocal, whereas the conformable derivative is local. Asıf Yokuş et al. [17] recently compared the Caputo and conformable formulations and found that, over a limited parameter range, they display broadly similar behavior with only minor differences—differences that stem from the distinct memory properties of local versus nonlocal operators.

To solve fractional-order differential equations (FODEs), researchers use a variety of methods, including the variational iteration method, homotopy perturbation and homotopy analysis methods, the exp-function method, the Adomian decomposition method, adaptive finite elements, sinc-collocation, and the residual power series method (RPSM). Surveys of commonly used techniques can be found in [18–25]. RPSM has been successfully applied to many linear and nonlinear models. In 2020, it was combined with the Laplace transform to form the Laplace residual power series method (LRPSM) [26,27].

In this work, we further develop RPSM by pairing it with the ARA transform (ARAT) [28–32], yielding the ARA residual power series method (ARA-RPSM). In the literature, numerous integral transforms like the Laplace, Fourier, Sumudu, and ARA transforms have been developed to address differential equations. Among these, the ARA transform, introduced in 2020, has emerged as a powerful method for solving fractional-order differential equations and systems. One of its notable advantages lies in its ability to handle differential equations with singular points near zero. Additionally, the ARA transform proves effective in solving specific functions where the Laplace transform is not applicable, as highlighted in [28]. This approach is fast, uses little memory, and is less sensitive to round-off errors.

The time-fractional fifth-order KdV equation generalizes the classical fifth-order Korteweg-de Vries (KdV) model by introducing a fractional time derivative. Instead of the usual first-order time derivative, it uses a derivative of fractional order, enabling a more faithful description of shallow-water and other nonlinear wave phenomena, especially in media with memory-dependent behavior.

Numerous studies have focused on KdV-type equations owing to their significance. For example, [33] presented analytical and numerical solutions to the fifth-order KdV equation; [34] proposed hyperelliptic solutions for certain modified KdV equations; and [35] reported a numerical investigation of the stochastic KdV equation. To address the generalized Kawahara equation, [36] introduced an operational matrix approach, while [37] employed a finite-difference method for the fractional KdV equation.

In this article, we apply the ARA residual power series method (ARA-RPSM) to obtain an approximate solution of the time-fractional KdV equation. This powerful approach provides a new series-based scheme for approximating the solution of a nonlinear fractional-order partial differential equation. The series

***Received**: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025*

https://doi.org/10.54361/ajmas.258374

coefficients can be computed rapidly using a limit-at-infinity argument, which reduces the time and effort required compared with other techniques. The proposed ARA–residual power series method (ARA-RPSM) is implemented here to get an approximate solution of the nonlinear homogeneous time fractional KdV equation of the form

$$D_t^{\propto} u + \epsilon u u_x + \rho u_{xxx} + \tau u_{xxxxx} = 0 \tag{1}$$

Subject to the initial condition (IC)

$$u(x,0) = e^x \tag{2}$$

Where

- $u(x,t)$ is the wave amplitude.
- $D_t^{\propto} u = \frac{\partial^{\propto} u}{\partial t^{\propto}}$ is the CFD of a fractional-order $\propto$ with $0 < \propto \leq 1$.
- $u_x = \frac{\partial u}{\partial x}$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$, and $u_{xxxxx} = \frac{\partial^5 u}{\partial x^5}$ are the first, third, and fifth spatial derivatives, respectively.
- ϵ, ρ, τ are nonzero parameters

This article is organized as follows. Section 'Basic definitions and theorems' reviews essential concepts, definitions, and results related to conformable derivative, fractional power series, and the ARA transform. Section 'Methodology of the ARA-RPSM' implements the ARA-RPSM to construct and predict solutions of the nonlinear time-fractional fifth-order KdV equation. Section 'Conformable approximate solution' evaluates the capability, simplicity, and efficiency of the proposed method by solving an example. Then the discussions on the obtained findings and main conclusion are demonstrated.

Basic definitions and theorems

In this section. we present the definition of the conformable fractional derivative. Also, the definition of the ARA transform, some properties, and theorems related to the fractional ARA-RPSM are revisited.

Definition 1. [16] Given a function $f : [0,\infty) \to \mathbb{R}$ then the "conformable fractional derivative" of the f of order $0 < \propto \leq 1$ is defined by:

$$f^{\propto}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\propto}) - f(t)}{\varepsilon}.$$

for all $t > 0$, and $\lim_{t\to 0} f^{\propto}(t)$ exists then defined $f^{\propto}(0) = \lim_{t\to 0} f^{\propto}(t)$.

Definition 2. [28].The ARA transform of order n of the continuous function $u(x,t)$ on the interval $I \times [0,\infty]$ for the variable t , is defined by

$$G_n[u(x,t)] = r \int_0^{\infty} t^{n-1} e^{-rt} u(x,t)\, dt \quad , r > 0.$$

In the following arguments, we present some basic properties of the ARA transform [23] that are essential in our research.

Properties of the ARA-transform [28-30]

Let $u(x,t)$ and $l(x,t)$ be continuous functions on $I \times [0,\infty)$ in which the ARA-transform for the variable t exists. Then we have:

(1) $G_n[au(x,t) + bl(x,t)] = aG_n[u(x,t)] + bG_n[l(x,t)]$, where a and b are nonzero constants.

(2) $\lim_{r\to\infty} G_1[u(x,t)] = u(x,0), \quad x \in I, r > 0.$

(3) $G_1[D_t^{\propto} u(x,t)] = r^{\propto} G_1[u(x,t)] - r^{\propto} u(x,0), \quad 0 < \propto \leq 1,\ x \in I, r > 0.$

(4) $G_2[t^{\propto}] = \frac{\Gamma(\propto+2)}{r^{\propto+1}}, \quad \propto > 0, r > 0.$

(5) $G_2[D_t^{\propto} u(x,t)] = r^{\propto} G_2[u(x,t)] - \propto r^{\propto-1} G_1[u(x,t)] + (\propto - 1) r^{\propto-1} u(x,0), \quad 0 < \propto \leq 1, x \in I, r > 0.$

(6) $G_2[D_t^{2\propto} u(x,t)] = r^{2\propto} G_2[u(x,t)] - 2 \propto r^{2\propto-1} G_1[u(x,t)] + (2\propto - 1) r^{2\propto-1} u(x,0) + (\propto - 1) r^{\propto-1} D_t^{\propto} u(x,0), \quad 0 < \propto \leq 1, x \in I, r > 0.$

(7) $\lim_{r\to\infty} r G_2[u(x,t)] = u(x,0), \quad x \in I,\ r > 0.$

***Received**: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025*

Theorem 1 [30]. Suppose that the fractional power series (FPS) representation of the function the form $u(x,t)$ at $t=0$ has the form

$$u(x,t)=\sum_{n=0}^{\infty} a_n(x)t^{n\propto},\qquad m-1<\propto\le m,\qquad m=1,2,\dots,\ \ 0\le t\le\beta.$$

If $u(x,t)$ and $D_t^{n\propto}u(x,t)$ are continuous on $I\times[0,\infty)$, Then the coefficients $a_n(x)$ have the form

$$a_n(x)=\frac{D_t^{n\propto}u(x,0)}{\Gamma(n\propto+1)},\quad \text{for } n=0,1,2,\dots \text{ where } D_t^{n\propto}=D_t^{\propto}.D_t^{\propto}\dots D_t^{\propto}\ \text{(n- times)}.$$

Theorem 2 [30]. Let $u(x,t)$ be a continuous function on $I\times[0,\beta]$ in which the ARA-transform for the variable t exists and has the FPS representation

$$G_2[u(x,t)]=\sum_{n=0}^{\infty}\frac{l_n(x)}{r^{n\propto+1}}\qquad,\qquad 0<\propto\le 1,\qquad x\in I \text{ and } r>0. \tag{3}$$

Then

$$l_n(x)=(n\propto+1)\,D_t^{n\propto}u(x,0). \tag{4}$$

Remark 1.

i. The k^{th} The truncated series of the series representation (3) is defined as follows

$$G_2[u(x,t)]_k=\sum_{n=0}^{k}\frac{l_{n(x)}}{r^{n\propto+1}}. \tag{5}$$

ii. If the ARA-transform of order two of the function $u(x,t)$ has the series representation (3), Then the ARA-transform of order one can be expressed as follows:

$$G_1[u(x,t)]=\sum_{n=0}^{\infty}\frac{l_n(x)}{(n\propto+1)r^{n\propto}}, \tag{6}$$

and the $k^{th}The$ truncated series is defined as follows:

$$G_1[u(x,t)]_k=\sum_{n=0}^{k}\frac{l_n(x)}{(n\propto+1)r^{n\propto}}\qquad. \tag{7}$$

iii. The inverse of the ARA-transform of order two for the fractional power series (3) is

$$u(x,t)=G_2^{-1}\left[\sum_{n=0}^{\infty}\frac{l_{n(x)}}{r^{n\propto+1}}\right](t)=\sum_{n=0}^{\infty}\frac{D_t^{n\propto}u(x,0)}{\Gamma(n\propto+1)}t^{n\propto}.$$

Theorem 3 [30]. Let $u(x,t)$ be a continuous function on $I\times[0,\beta]$ in which the ARA-transform for the variable t exists.

Assume that $G_1[u(x,t)]$ has the following series representation:

$$G_1[u(x,t)]=\sum_{n=0}^{\infty}\frac{c_n(x)}{r^{n\propto}}$$

where

$$c_n(x)=D_t^{n\propto}u(x,0).$$

If $\left|G_1\left[D_t^{(n+1)\propto}u(x,t)\right]\right|\le M$ on $0<r\le d$, then the remainder $R_n(x,r)$ Satisfies the following inequality:

$$|R_n(x,r)|\le\frac{M(x)}{r^{(n+1)\propto}}\ ,\qquad x\in I,\qquad 0<r\le d.$$

Methodology of the ARA-RPSM

Received: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025

In this section, we present the methodology of ARA-RPSM for solving the time-dependent fifth-order Korteweg-de Vries (fKdV) equation. The main idea of the proposed method is based on applying the ARA-transform on the given equation and using Taylor's expansion to create solitary solutions.
To perform the ARA-RPSM, consider the nonlinear homogeneous time fractional KdV equation (1) when $\epsilon = 1,\ \rho = -1, \tau = 1$.
Operate the ARA-transform of order two G_2 with respect to the variable t, on both sides of equation (1), we get

$$G_2[D_t^\alpha u(x,t)] = G_2\big[N[u(x,t)]\big] + G_2[R[u(x,t)]]. \tag{8}$$

Dy to the initial condition:

$$u(x,0) = l_0(x) \tag{9}$$

Here, $D_t^\alpha u(x,t)$ represents the conformable derivative of $u(x,t)$, $N[u(x,t)]$ and $R[u(x,t)]$ denote nonlinear and linear terms of Eq. (1), respectively.
Using property 6 and IC (9), Eq. (8) becomes.

$$r^\alpha G_2[u(x,t)] - \alpha r^{\alpha-1} G_1[u(x,t)] + (\alpha-1) r^{\alpha-1} u(x,0) - G_2\big[N\big(G_2^{-1}[G_2[u(x,t)]]\big) - G_2\big[R\big(G_2^{-1}[G_2[u(x,t)]]\big)\big]\big] = 0. \tag{10}$$

Assume that the **ARA-RPS** solution of Eq.(10) has the following series representations.

$$G_1[u(x,t)] = \sum_{n=0}^{\infty} \frac{l_n(x)}{(n\alpha+1) r^{n\alpha}}, \tag{11}$$

$$G_2[u(x,t)] = \sum_{n=0}^{\infty} \frac{l_n(x)}{r^{n\alpha+1}}. \tag{12}$$

Using the fact in property 7.

$$\lim_{r\to\infty} r G_2 [u(x,t)] = u(x,0),$$

We have $l_n(x) = a(x)$. Hence, the series representation (12) becomes.

$$G_2[u(x,t)] = \frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}} + \sum_{n=2}^{\infty} \frac{l_n(x)}{r^{n\alpha+1}} \tag{13}$$

To find $l_1(x)$, we multiply both sides of Eq.(13) by $r^{\alpha+1}$ and take the limit as $r \to \infty$ to obtain.

$$\lim_{r\to\infty} r^{\alpha+1} G_2[u(x,t)] = \lim_{r\to\infty} r^\alpha l_0 + l_1(x) + \lim_{r\to\infty} \sum_{n=2}^{\infty} \frac{l_n(x)}{r^{n\alpha+1}},$$

Which is equivalent to.

$$l_1(x) = \lim_{r\to\infty} r\big(r^\alpha G_2[u(x,t)] - r^{\alpha-1} l_0(x)\big).$$

Property 5 yields that.

$$\begin{aligned} l_1(x) &= \lim_{r\to\infty} r\big(G_2[D_t^\alpha u(x,t)] + \alpha r^{\alpha-1} G_1[u(x,t)] - \alpha r^{\alpha-1} l_0(x)\big) \\ &= \lim_{r\to\infty} r G_2[D_t^\alpha u(x,t)] + \lim_{r\to\infty} \alpha\big(r^\alpha G_1[u(x,t)] - r^\alpha l_0(x)\big). \end{aligned}$$

Using property 3, we get.

$$l_1(x) = \lim_{r\to\infty} r G_2[D_t^\alpha u(x,t)] + \alpha \lim_{r\to\infty} G_1[D_t^\alpha u(x,t)].$$

Properties 2 and 7 lead to.

$$l_1(x) = (\alpha+1) D_t^\alpha l_0(x).$$

Thus, the ARA-RPS solution of Eq. (10) has the series representations.

$$G_1[u(x,t)] = l_0(x) + \frac{l_1(x)}{r^\alpha} + \sum_{n=2}^{\infty} \frac{l_n(x)}{(n\alpha+1) r^{n\alpha}}, \tag{14}$$

$$G_2[u(x,t)] = \frac{l_0}{r} + \frac{(\alpha+1) l_1(x)}{r^{\alpha+1}} + \sum_{n=2}^{\infty} \frac{l_n(x)}{r^{n\alpha+1}}. \tag{15}$$

and the k^{th} *The* truncated series expansion of Eq. (14) and Eq. (15) have the

$$G_1[u(x,t)]_k = l_0(x) + \frac{l_1(x)}{r^\alpha} + \sum_{n=2}^{k} \frac{l_n(x)}{(n\alpha+\alpha 1) r^{n\alpha}}, \tag{16}$$

$$G_2[u(x,t)]_k = \frac{l_0(x)}{r} + \frac{(\alpha+1) l_1(x)}{r^{\alpha+1}} + \sum_{n=2}^{k} \frac{l_n(x)}{r^{n\alpha+1}}. \tag{17}$$

To find the coefficients of the series expansions in Eqs. (16) and (17), we define the **ARA-**residual function of Eq. (10) as follows.

Received: *19-06-2025* - ***Accepted***: *17-08-2025* - ***Published***: *21-08-2025*

$$G_2Res(x,r) = G_2[u(x,t)] - \frac{\alpha}{r}G_1[u(x,t)] + \frac{(\alpha-1)}{r}u(x,0) - \frac{1}{r^\alpha}G_2\left[N\left(G_2^{-1}[G_2[u(x,t)]]\right) - \frac{1}{r^\alpha}G_2\left[R\left(G_2^{-1}[G_2[u(x,t)]]\right)\right]\right] = 0. \quad (18)$$

and the k^{ht} ARA-residual function is.

$$G_2Res_k(x,r) = G_2[u(x,t)]_k - \frac{2\alpha}{r}G_1[u(x,t)]_k + \frac{(2\alpha-1)}{r}l_0(x) - \frac{1}{r^\alpha}G_2\left[N\left(G_2^{-1}[G_2[u(x,t)]]_k\right) - \frac{1}{r^\alpha}G_2\left[R\left(G_2^{-1}[G_2[u(x,t)]]_k\right)\right]\right], k = 2,3,\dots . \quad (19)$$

In order to find the coefficients $l_n(x), n \geq 2$ in the series expansion (17), multiply both sides of Eq.(19) by $r^{k\alpha+1}$, $k = 2,3,\dots$, and take the limit $as\ r \to \infty$, then solve the equations.

$$\lim_{r\to\infty} r^{k\alpha+1}\, G_2Res_k(x,r) = 0, \qquad k = 2,3,\dots,$$

The following facts are needed to obtain the ARA-RPS solution.

$$G_2Res(x,r) = 0, \qquad x \in I, \qquad r > 0.$$

$$\lim_{k\to\infty} G_2\, Res_k(x,r) = G_2Res(x,r), \quad x \in I, \quad r > 0.$$

$$\lim_{r\to\infty} r\, G_2Res(x,r) = 0, and \lim_{r\to\infty} r\, G_2\, Res_k(x,r) = 0, \quad x \in I, \quad r > 0.$$

$$\lim_{r\to\infty} r^{k\alpha+1}\, G_2\, Res(x,r) = \lim_{r\to\infty} r^{k\alpha+1}\, G_2\, Res_k(x,r) = 0, \quad x \in I, r > 0.$$

The obtained coefficients $l_n(x)$ are substituted in the series solution (12), then operate the inverse ARA transform of order two G_2^{-1} to get the solution of the IVP (1) and (2) in the original space.

Conformable approximate solution for fKdV equation using ARA-RPSM

Considering the time-fractional fKdV equation as follows [39]:

$$D_t^\alpha u(x,t) + u(x,t)\, u_x(x,t) - u(x,t)\, u_{xxx}(x,t) + u_{xxxxx}(x,t) = 0\,, \qquad 0 < \alpha \leq 1 \quad (20)$$

Initial Condition

$$u(x,0) = e^x. \quad (21)$$

The exact solution for Eq. (20) when α =1 is [38]

$$u(x,t) = e^{x-t}.$$

Operating ARA- of order two on Eq.(20)

$$G_2[D_t^\alpha u(x,t)] + G_2[G_2^{-1}[G_2u(x,t)]\frac{\partial}{\partial x}G_2^{-1}[G_2u(x,t)]] - G_2[G_2^{-1}[G_2u(x,t)]\frac{\partial^3}{\partial x^3}G_2^{-1}[G_2u(x,t)]] + G_2[\frac{\partial^5}{\partial x^5}G_2^{-1}[G_2u(x,t)]] = 0. \quad (22)$$

Which corresponds to r

$$r^\alpha G_2[u(x,t)] - \alpha r^{\alpha-1}G_1[u(x,t)] + (\alpha-1)r^{\alpha-1}u(x,0) + G_2\left[G_2^{-1}[G_2[u(x,t)]]\frac{\partial}{\partial x}G_2^{-1}[G_2[u(x,t)]]\right] - G_2[G_2^{-1}[G_2[u(x,t)]]\frac{\partial^3}{\partial x^3}G_2^{-1}[G_2[u(x,t)]]] + G_2\left[\frac{\partial^5}{\partial x^5}G_2^{-1}[G_2[u(x,t)]]\right] = 0. \quad (23)$$

Simplifying Eq. (23), we have.

$$G_2[u(x,t)] - \frac{\alpha}{r}G_1[u(x,t)] + \frac{\alpha-1}{r}l_0(x) + \frac{1}{r^\alpha}G_2\left[G_2^{-1}[G_2[u(x,t)]]\frac{\partial}{\partial x}G_2^{-1}[G_2[u(x,t)]]\right] - \frac{1}{r^\alpha}G_2\left[G_2^{-1}[G_2[u(x,t)]]\frac{\partial^3}{\partial x^3}G_2^{-1}[G_2[u(x,t)]]\right] + \frac{1}{r^\alpha}G_2\left[\frac{\partial^5}{\partial x^5}G_2^{-1}[G_2[u(x,t)]]\right]. \quad (24)$$

Consider expanding the ARA-RPS of Eq. (24) as follows:

$$G_1[u(x,t)](r) = \sum_{n=0}^{\infty}\frac{l_n(x)}{(n\alpha+1)r^{n\alpha}}\,, \quad (25)$$

$$G_2[u(x,t)](r) = \sum_{n=0}^{\infty}\frac{l_n(x)}{r^{n\alpha+1}}. \quad (26)$$

And the j^{th} truncates the series of the expansions (25) and (26) are.

$$G_1[u(x,t)]_j(r) = \sum_{n=0}^{j}\frac{l_n(x)}{(n\alpha+1)r^{n\alpha}}\,, \quad (27)$$

Received: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025

https://doi.org/10.54361/ajmas.258374

$$G_2[u(x,t)]_j(r) = \sum_{n=0}^{j} \frac{l_n(x)}{r^{n\alpha+1}}. \tag{28}$$

By taking the limit as $r \to \infty$, after multiplying both side of Eq.(28) by r, we get.

$$\lim_{r\to\infty} r\, G_2[u(x,t)]_j(r) = l_0(x) + \lim_{r\to\infty} \sum_{n=1}^{j} \frac{l_n(x)}{r^{n\alpha+1}}.$$

Using the fact.

$$\lim_{r\to\infty} r\, G_2[u(x,t)]_j(r) = u(x,0)\,,$$

And the initial condition in Eq.(21), we get.

$$l_0(x) = e^x.$$

Hence, from the series representations (8) and (9), we get.

$$G_1[u(x,t)]_j(r) = e^x + \sum_{n=1}^{j} \frac{l_n(x)}{(n\alpha+1)r^{n\alpha}}\,, \tag{29}$$

$$G_2[u(x,t)]_j(r) = \frac{e^x}{r} + \sum_{n=1}^{j} \frac{l_n(x)}{r^{n\alpha+1}}. \tag{30}$$

The ARA-Residual function of Eq. (24) in now given by:

$$\begin{aligned} G_2Res(x,r) = {} & G_2[u(x,t)] - \frac{\alpha}{r} G_1[u(x,t)] + \frac{\alpha-1}{r} e^x + \frac{1}{r^\alpha} G_2\left[G_2^{-1}[G_2[u(x,t)]] \frac{\partial}{\partial x} G_2^{-1}[G_2[u(x,t)]]\right] \\ & - \frac{1}{r^\alpha} G_2\left[G_2^{-1}[G_2[u(x,t)]] \frac{\partial^3}{\partial x^3} G_2^{-1}[G_2[u(x,t)]]\right] \\ & + \frac{1}{r^\alpha} G_2\left[\frac{\partial^5}{\partial x^5} G_2^{-1}[G_2[u(x,t)]]\right], \end{aligned} \tag{31}$$

And the j^{th}ARA-Residual function of Eq. (12) is.

$$\begin{aligned} G_2Res_j(x,r) = {} & G_2[u(x,t)]_j - \frac{\alpha}{r} G_1[u(x,t)]_j + \frac{\alpha-1}{r} e^x + \frac{1}{r^\alpha} G_2\left[G_2^{-1}\left[G_2[u(x,t)]_j\right] \frac{\partial}{\partial x} G_2^{-1}\left[G_2[u(x,t)]_j\right]\right] \\ & - \frac{1}{r^\alpha} G_2\left[G_2^{-1}\left[G_2[u(x,t)]_j\right] \frac{\partial^3}{\partial x^3} G_2^{-1}\left[G_2[u(x,t)]_j\right]\right] \\ & + \frac{1}{r^\alpha} G_2\left[\frac{\partial^5}{\partial x^5} G_2^{-1}\left[G_2[u(x,t)]_j\right]\right]. \end{aligned} \tag{32}$$

To determent the first unknown coefficient $l_1(x)$ in Eq. (29) and Eq. (30) we substituting $G_1[u(x,t)]_1(r)$ and $G_2[u(x,t)]_1(r)$ into $G_2Res_1(x,r)$ to obtain.

$$\begin{aligned} G_2Res_1(x,r) = {} & G_2[u(x,t)]_1 - \frac{\alpha}{r} G_1[u(x,t)]_1 + \frac{\alpha-1}{r} e^x + \frac{1}{r^\alpha} G_2\left[G_2^{-1}[G_2[u(x,t)]_1] \frac{\partial}{\partial x} G_2^{-1}[G_2[u(x,t)]_1]\right] \\ & - \frac{1}{r^\alpha} G_2\left[G_2^{-1}[G_2[u(x,t)]_1] \frac{\partial^3}{\partial x^3} G_2^{-1}[G_2[u(x,t)]_1]\right] + \frac{1}{r^\alpha} G_2\left[\frac{\partial^5}{\partial x^5} G_2^{-1}[G_2[u(x,t)]_1]\right]. \end{aligned} \tag{33}$$

Substitute

$$G_1[u(x,t)]_1 = l_0(x) + \frac{l_1(x)}{(\alpha+1)r^{\alpha+1}}\,,$$

And

$$G_2[u(x,t)]_1 = \frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}.$$

In Eq. (33) after simple computations, we have.

$$\begin{aligned} G_2Res_1(x,r) = {} & \left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right) - \frac{\alpha}{r}\left(l_0(x) + \frac{l_1(x)}{(\alpha+1)r^{\alpha+1}}\right) + \frac{\alpha-1}{r} e^x + \frac{1}{r^\alpha} G_2\left[G_2^{-1}\left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right) \frac{\partial}{\partial x} G_2^{-1}\left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right)\right] \\ & - \frac{1}{r^\alpha} G_2\left[G_2^{-1}\left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right) \frac{\partial^3}{\partial x^3} G_2^{-1}\left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right)\right] + \frac{1}{r^\alpha} G_2\left[\frac{\partial^5}{\partial x^5} G_2^{-1}\left(\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}}\right)\right]. \end{aligned} \tag{34}$$

Thus,

$$\begin{aligned} G_2Res_1(x,r) = {} & \frac{l_1(x)}{(\alpha+1)r^{\alpha+1}} + \frac{l_0(x)l_0^{'}(x)}{r^{\alpha+2}} + \frac{l_0(x)l_1^{'}(x)}{r^{2\alpha+2}} + \frac{l_1(x)l_0^{'}(x)}{r^{2\alpha+2}} + \frac{l_1(x)\, l_1^{'}(x)}{r^{3\alpha+2}} - \frac{l_0(x)l_0^{(3)}(x)}{r^{\alpha+2}} - \frac{l_0(x)l_1^{(3)}(x)}{r^{2\alpha+2}} \\ & + \frac{l_1(x)l_0^{(3)}(x)}{r^{2\alpha+1}} + \frac{l_1(x)l_1^{(3)}(x)}{r^{3\alpha+2}} + \frac{l_0^{(5)}(x)}{r^{\alpha+1}} + \frac{l_1^{(5)}(x)}{r^{2\alpha+1}}. \end{aligned} \tag{35}$$

***Received**: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025*

https://doi.org/10.54361/ajmas.258374

By taking the limit as $r \to \infty$, after multiplying Eq.(35) by $r^{\alpha+1}$, the fact $\lim_{r\to\infty}(r^{\alpha+1} G_2 Res_1(r,x)) = 0$, yields that.

$$l_1(x) = (\alpha + 1)\, l_0^{(5)}.$$

Therefore,

$$l_1(x) = (\alpha + 1)\, x^{5-\alpha} e^x \,. \tag{36}$$

Similarly, to find $l_2(x)$, we Substitute

$$G_1[u(x,t)]_2(r) = l_0(x) + \frac{l_1(x)}{(\alpha+1)r^{\alpha}} + \frac{l_2(x)}{(2\alpha+1)r^{2\alpha}}\,, \tag{37}$$

And

$$G_2[u(x,t)]_2(r) = \frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}} + \frac{l_2(x)}{r^{2\alpha+1}} \tag{38}$$

Into $G_2 Res_2(x,r)$ and solve the equation $\lim_{r\to\infty} r^{2\alpha+1} G_2 Res_2(x,r) = 0$.

To get:

$$G_2 Res_2(x,r) = \left[\frac{l_0(x)}{r} + \frac{l_1(x)}{r^{\alpha+1}} + \frac{l_2(x)}{r^{2\alpha+1}}\right] - \frac{\alpha}{r}\left[l_0(x) + \frac{l_1(x)}{(\alpha+1)r^{\alpha}} + \frac{l_2(x)}{(2\alpha+1)r^{2\alpha}}\right] + \left[\frac{\alpha-1}{r} e^x\right]$$
$$+ \left[\frac{l_0(x)}{r^{\alpha+1}} + \frac{l_1(x)}{r^{2\alpha+1}} + \frac{l_2(x)}{r^{3\alpha+1}}\right]\left[\frac{l_0'(x)}{r} + \frac{l_1'(x)}{r^{\alpha+1}} + \frac{l_2'(x)}{r^{2\alpha+1}}\right] - \left[\frac{l_0(x)}{r^{\alpha+1}} + \frac{l_1(x)}{r^{2\alpha+1}} + \frac{l_2(x)}{r^{3\alpha+1}}\right]\left[\frac{l_0^{(3)}(x)}{r} + \frac{l_1^{(3)}(x)}{r^{\alpha+1}} + \frac{l_2^{(3)}(x)}{r^{2\alpha+1}}\right]$$
$$+ \left[\frac{l_0^{(5)}(x)}{r} + \frac{l_1^{(5)}(x)}{r^{\alpha+1}} + \frac{l_2^{(5)}(x)}{r^{2\alpha+1}}\right]. \tag{39}$$

Upon multiplying Eq. (39) by $r^{2\alpha+1}$, and letting r tend to zero, it follows that:

$$l_2(x) = \frac{2\alpha+1}{\alpha+1}\, l_1^{(5)}(x). \tag{40}$$

By using conformable definition:

$$l_2(x) = (2\alpha+1)\big[[(5-\alpha)!\, x^{5-2\alpha} e^x] + 5[(5-\alpha)(4-\alpha)(3-\alpha)(2-\alpha)x^{6-2\alpha} e^x] + 10[(5-\alpha)(4-\alpha)(3-\alpha)x^{7-2\alpha} e^x]$$
$$+ 11[(5-\alpha)(4-\alpha)x^{8-2\alpha} e^x] + 6[(5-\alpha)x^{9-2\alpha} e^x]\big]. \tag{41}$$

Repeating the same arguments as before, we get the solution of Eq.(23) as:

$$G_2[u(x,t)] = \frac{e^x}{r} + \frac{\alpha+1}{r^{\alpha+1}}\, x^{5-\alpha} e^x$$
$$+ \frac{(2\alpha+1)}{r^{2\alpha+1}}\Big[\big[[(5-\alpha)!\, x^{5-2\alpha} e^x] + 5[(5-\alpha)(4-\alpha)(3-\alpha)(2-\alpha)x^{6-2\alpha} e^x]$$
$$+ 10[(5-\alpha)(4-\alpha)(3-\alpha)x^{7-2\alpha} e^x] + 11[(5-\alpha)(4-\alpha)x^{8-2\alpha} e^x] + 6[(5-\alpha)x^{9-2\alpha} e^x]\big]. \Big]. \tag{42}$$

Applying the inverse ARAT on Eq. (42), the solution of problems (20) and (21) is obtained as follows:

$$u(x,t) = e^x + \frac{\alpha+1}{\Gamma(\alpha+1)}\, t^{\alpha} x^{5-\alpha} e^x$$
$$+ \frac{2\alpha+1}{\Gamma(2\alpha+1)}\, t^{2\alpha+1}\Big[\big[[(5-\alpha)!\, x^{5-2\alpha} e^x] + 5[(5-\alpha)(4-\alpha)(3-\alpha)(2-\alpha)x^{6-2\alpha} e^x]$$
$$+ 10[(5-\alpha)(4-\alpha)(3-\alpha)x^{7-2\alpha} e^x] + 11[(5-\alpha)(4-\alpha)x^{8-2\alpha} e^x] + 6[(5-\alpha)x^{9-2\alpha} e^x]\big]\Big] + \cdots \tag{43}$$

Results and Discussion

The ARA- RPSM solution u(x,t) is illustrated in Figure 2, for $0 \le x \le 10$ and $0 \le t \le 1$ when α =0.2, α =0.5, α = 0.8, α=1. When α=1 is chosen among the different values of α, the u(x,t) is closest to the exact solution as in Figure 1. Here, we observe that the ARA-RPSM solution converges rapidly with increasing order of approximation. Furthermore, from (figures 1and 2), it is evident that the RPS results are nearly identical to the numerical results. The results are very consistent with the increasing time. Three-dimensional surface graphs are used to illustrate the dynamical behavior of the earned results. Through graphical illustrations, it can be noticed that various forms of traveling wave structures are obtained for the time fractional nonlinear fKdV using the ARA residual power series method method.

In this study, the ARA-RPSMARA-RPSMARA-RPSM was utilized to gain an approximate solution of the time fractional fKdV equation. In the reliability of the proposed method for the time fractional fKdV equation had emerged. Besides, the third ARA-RPS solutions were demonstrated by 3D graphs. It could be seen in Figure 2. All graphics were shown by the help of MATLAB. In addition, it was seen that ARA-RPSMARA-RPSM achieved a high accuracy when the numerical results were analyzed in this paper.

Received: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025

Alqalam Journal of Medical and Applied Sciences. 2025;8(3):1785-1794
https://doi.org/10.54361/ajmas.258374

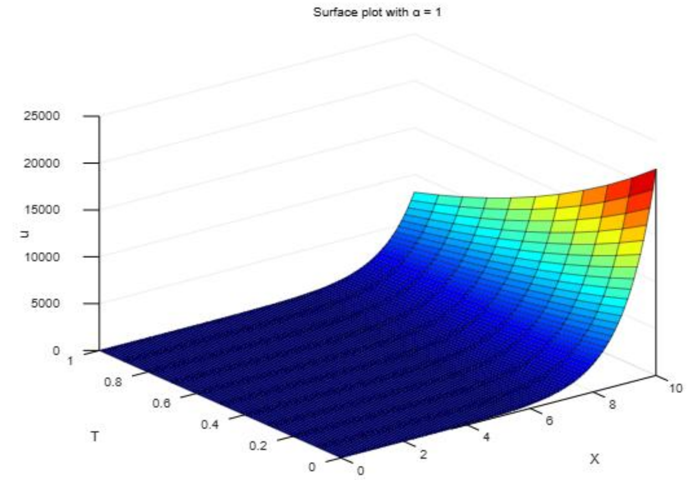

Figure1. Three-dimensional surface graph of the exact solution of $u(x,t)$ ***when*** $\mathbf{0 \le x \le 10, 0 \le t \le 1}$, $\boldsymbol{\alpha = 1}$.

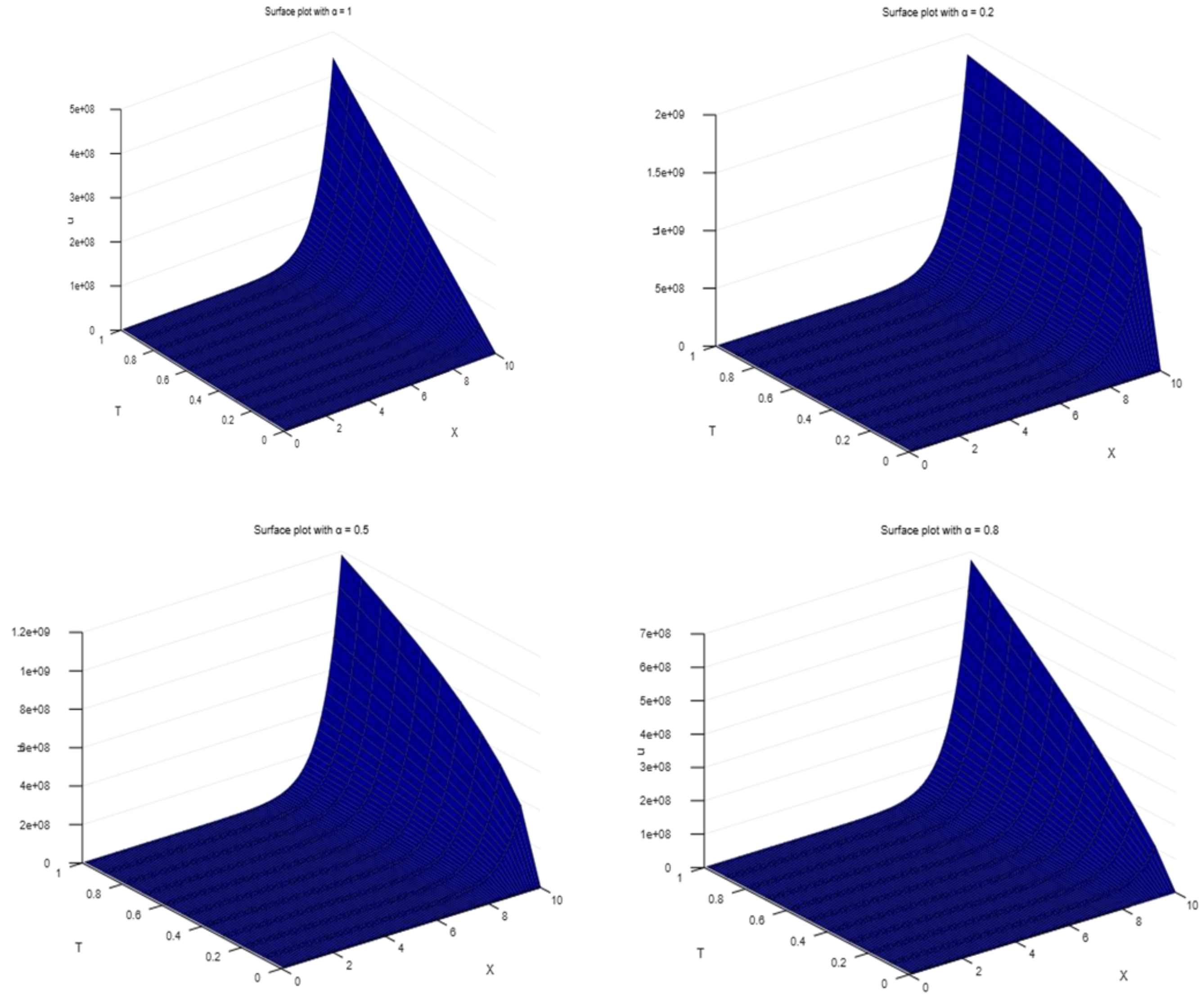

Figure1. Three-dimensional surface graphs of $u_3(x,t)$ ***when*** $\mathbf{0 \le x \le 10, 0 \le t \le 1}$, $\boldsymbol{\alpha = 0.2, 0.5, 0.8, 1}$.

Conclusion

In this study, the ARA residual power series method was introduced to obtain an approximate analytical solution for time-fractional Korteveg de Vries (KdV) in a conformable sense. Numerical results and comparison with the exact solution show that the present method is a very powerful and reliable technique

Received*: 19-06-2025 -* ***Accepted****: 17-08-2025 -* ***Published****: 21-08-2025*

and producing highly approximate results. Compared to other techniques, the method is very simple to apply without linearization, perturbation, or discretization or any transformations. Also, it is a good tool to use to calculate the approximate solutions of a wide range of fractional partial differential equations.

Acknowledgments
The authors would like to thank the referees for their useful suggestions and comments, which have improved this paper.

Conflicts of Interest
The authors declare that they have no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper. The authors declare that there are no conflicts of interest.

References

1. Adomian G. A review of the decomposition method in applied mathematics. J Math Anal Appl. 1988;135(2):501-544.
2. Akgul A. A novel method for a fractional derivative with non-local and non-singular kernel. Chaos Solitons Fractals. 2018;114:478-482.
3. Akgul EK. Solutions of the linear and nonlinear differential equations within the generalized fractional derivatives. Chaos. 2019;29(2):023108.
4. Atangana A. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. Chaos Solitons Fractals. 2017;102:396-406.
5. Attia RAM, Xia Y, Zhang X, Khater MMA. Analytical and numerical investigation of soliton wave solutions in the fifth-order KdV equation within the KdV-KP framework. Results Phys. 2023;51:106646.
6. Burqan A. A novel scheme of the ARA transform for solving systems of partial fractional differential equations. Fractal Fractional. 2023;7(2):306.
7. Burqan A, Saadeh R, Qazza A. A novel numerical approach in solving fractional neutral pantograph equations via the ARA integral transform. Symmetry. 2022;14(1):50.
8. Burqan A, Saadeh R, Qazza A, Momani S. ARA residual power series method for solving partial fractional differential equations. Alexandria Eng J. 2023;62:47-62.
9. Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Prog Fract Differ Appl. 2015;1(2):73-85.
10. Cifani S, Jakobsen E. Entropy solution theory for fractional degenerate convection-diffusion equations. Ann I H Poincaré Anal Non Linéaire. 2011;28(3):413-441.
11. D'Ambrosio R, Di Giovacchino S. Numerical conservation issues for the stochastic Korteweg–de Vries equation. J Comput Appl Math. 2023;424:114967.
12. Das S. Analytical solution of a fractional diffusion equation by variational iteration method. Comput Math Appl. 2009;57(3):483-487.
13. Dwivedi M, Sarkar T. Fully discrete finite difference schemes for the fractional Korteweg-de Vries equation. J Sci Comput. 2024;101(1):30.
14. El-Ajou A. Adapting the Laplace transform to create solitary solutions for the nonlinear time-fractional dispersive PDEs via a new approach. Eur Phys J Plus. 2021;136(2):229.
15. El-Ajou A, Odibat Z, Momani S, Alawneh A. Construction of analytical solutions to fractional differential equations using homotopy analysis method. Int J Appl Math. 2010;40(2).
16. Eriqat T, El-Ajou A, Moa'ath N, Al-Zhour Z, Momani S. A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations. Chaos Solitons Fractals. 2020;138:109957.
17. Hasan S, Al-Smadi M, El-Ajou A, Momani S, Hadid S, Al-Zhour Z. Numerical approach in the Hilbert space to solve a fuzzy Atangana-Baleanu fractional hybrid system. Chaos Solitons Fractals. 2021;143:110506.
18. He HJ. Approximate analytical solution for seepage flow with fractional derivatives porous media. Comput Methods Appl Mech Eng. 1998;167(1-2):57-68.
19. Ismail G, Abdl-Rahim H, Ahmad H, Chu Y. Fractional residual power series method for the analytical and approximate studies of fractional physical phenomena. Open Phys. 2020;18(1):799-805.
20. Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results. Comput Math Appl. 2006;51(9-10):1367-1376.
21. Kaya D. An Explicit and Numerical Solutions of Some Fifth Order KdV Equation by Decomposition Method. Appl Math Comput. 2003;144(2-3):353-363.
22. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math. 2014;264:65-70.
23. Khandaqji M, Burqan A. Results on sequential conformable fractional derivatives with applications. J Comput Anal Appl. 2021;29(6):1115-1125.
24. Li ZB. An extended fractional complex transform. Int J Nonlinear Sci Numer Simul. 2010;11(5):335-338.
25. Magin RL, Ingo C, Colon-Perez L, Triplett W, Mareci TH. Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy. Microporous Mesoporous Mater. 2013;178:39-43.
26. Mainardi F, Raberto M, Gorenflo R, Scalas E. Fractional calculus and continuous-time finance II: the waiting-time distribution. Physica A. 2000;287(3-4):468-481.
27. Matsutani S. On real hyperelliptic solutions of focusing modified KdV equation. Math Phys Anal Geom. 2024;27(1):19.

***Received**: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025*

https://doi.org/10.54361/ajmas.258374

28. Miller K, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley; 1993.
29. Momani S, Odibat Z. Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations. Comput Math Appl. 2007;54(7-8):910-919.
30. Oldham K, Spanier J. The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. New York: Elsevier; 1974.
31. Qazza A, Burqan A, Saadeh R. A new attractive method in solving families of fractional differential equations by a new transform. Mathematics. 2021;9(23):3039.
32. Sabatier J, Agrawal OP, Machado JAT, eds. Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Dordrecht: Springer; 2007.
33. Saadeh R, Qazza A, Burqan A. A new integral transform: ARA transform and its properties and applications. Symmetry. 2020;12(6):925.
34. Shqair M, El-Ajou A, Nairat M. Analytical solution for multi-energy groups of neutron diffusion equations by a residual power series method. Mathematics. 2019;7(7):633.
35. Ünal SÇ. Solutions of time fractional fKdV equation using the residual power series method. Cumhuriyet Sci J. 2022;43(3):468-476.
36. Wang Q. Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. Appl Math Comput. 2006;182(2):1048-1055.
37. Yokus A, Durur H, Kaya D, Ahmad H, Nofal TA. Numerical comparison of Caputo and conformable derivatives of time fractional Burgers-Fisher equation. Results Phys. 2021;25:104247.
38. Zhang S, Zhang H. Fractional sub-equation method and its applications to nonlinear fractional PDEs. Phys Lett A. 2011;375(7):1069-1073.
39. Ahmed HM, Hafez RM, Abd-Elhameed WM. A computational strategy for nonlinear time-fractional generalized Kawahara equation using new eighth-kind Chebyshev operational matrices. Phys Scr. 2024;99(4):045250

***Received**: 19-06-2025 - **Accepted**: 17-08-2025 - **Published**: 21-08-2025*