https://alqalam.utripoli.edu.ly/science/ eISSN 2707-7179

Original article

A Note on Fractional Double Natural Transform

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ARTICLE INFO

http://doi.org/10.5281/zenodo.4442629

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Received: *21-12-2020*
Accepted: *14-01-2021*
Published: *15-01-2021*

Keywords: *Modified Riemann Liouville derivative, double Natural transform, fractional double Laplace, double Sumudu transforms, convolution theorem.*

ABSTRACT

In this article, we present a definition of fractional double Natural transform of order α, $0 < \alpha \leq 1$, for fractional differentiable functions. Some essential properties of fractional double Natural transform are determined. Furthermore, we set a relation between fractional double Natural transform and fractional double Laplace, fractional double Sumudu transforms.

Cite this article: *Omran M. A Note on Fractional Double Natural Transform. Alq J Med App Sci. 2021;4(1):85-90.*

1. INTRODUCTION

The Natural transform was established by Khan and Khan in 2008 see [1]. More theoretical details and properties were studied by Belgacem and Silambarasan [2]. Recently, Kilicman and Omran [3] generalized the concept of one- dimensional Natural transform to two- dimensional Natural transform namely, double Natural transform, which used to the solutions of telegraphs, wave, and partial integro-differential equations.

The aims of this study are to introduce a definition of a fractional double Natural transform of order α, $0 < \alpha \leq 1$, for fractional differentiable function, and present the connection between this fractional transform and fractional double Laplace, fractional double Sumudu transforms.

1.1 *Double Integral Transforms*

Definition 1.1.1 [4] Double Laplace transform of a function $f(x,y)\ for\ x,y > 0$ is denoted by $\mathcal{L}^2\{f(x,y)\} = F^2(s,p)$, and defined as

$$\mathcal{L}^2\{f(x,y)\} = F^2(s,p) = \int_0^\infty \int_0^\infty e^{-sx}\, e^{-py}\, f(x,y)dxdy\,,$$

$$where\ x,y \in R_+. \quad (1.1.1)$$

Definition 1.1.2 [5] Double Sumudu transform of a function $f(x,y)$ is denoted by $S^2\{f(x,y)\} = G^2(u,v)$, and defined as

$$S^2\{f(x,y)\} = G^2(u,v) = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\frac{x}{u}} e^{-\frac{y}{v}} f(x,y)dxdy. \qquad (1.1.2)$$

Definition 1.1.3 [3] Let $f(x,y)$ be a function and $x, y \in R_+$, then double Natural transform is stated as

$$\mathbb{N}_+^2\{f(x,y)\} = \mathcal{R}_+^2[(s,p);(u,v)] = \int_0^\infty\int_0^\infty e^{-(sx+py)} f(ux,vy)\, dxdy. \qquad (1.1.3)$$

The above formula can be written in other form as

$$\mathbb{N}_+^2\{f(x,y)\} = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{sx}{u}+\frac{py}{v}\right)} f(x,y)\, dxdy. \qquad (1.1.4)$$

1.2 *Fractional Derivative via Fractional Difference*

Definition 1.2.1 [6] Let be a continuous function and not necessarily differentiable function, then the forward operator $FW(h)$ is defined as follows

$$FW(h)g(t) \coloneqq g(t+h),$$

where $h > 0$ denote a constant discretization span. Furthermore, the fractional difference of $g(t)$ is known as $\Delta^\alpha g(t) := (FW-h)^\alpha g(t) \coloneqq \sum_{n=0}^{\infty}(-1)^n \binom{\alpha}{n} g[t+(\alpha-n)h]$, where $0 < \alpha < 1$, and the α - derivative of $g(t)$ is known as

$$g^{(\alpha)}(t) = \lim_{h\downarrow 0}\frac{\Delta^\alpha g(t)}{h^\alpha},$$

For more details we refer to [7, 8]

1.3 *Modified Fractional Riemann-Liouville Derivative*

An alternative definition of the Riemann-Liouville fractional derivative was proposed as the follows.
Definition 1.3.1 [6] Let $g(t)$ be a continuous function, but not necessarily differentiable, then;

- Suppose that $g(t) = C$, where C is a constant, so α-derivative of the function $g(t)$ is

$$D_t^\alpha C = C\,\Gamma^{-1}(1-\alpha)t^{-\alpha}, \quad \alpha \le 0,$$
$$= 0,$$

otherwise.

- On the other hand, when $g(t) \neq C$ then we will put

$$g(t) = g(0) + \big(g(t) - g(0)\big),$$

and fractional derivative of the function $g(t)$ will be known as

$$g^\alpha(t) = D_t^\alpha g(0) + D_t^\alpha\big(g(t) - g(0)\big),$$

for negative α, $(\alpha < 0)$ one has

$$D_t^\alpha\big(g(t) - g(0)\big) = \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\eta)^{-\alpha-1} g(\eta)d\eta, \quad \alpha < 0,$$

whilst for positive α, we will set

$$D_t^\alpha\big(g(t) - g(0)\big) = D_t^\alpha g(t) = D_t\big(g^{(\alpha-1)}\big).$$

When $n \le \alpha < n+1$, we set

$$g^{(\alpha)}(t) \coloneqq \left(g^{(\alpha-n)}(t)\right)^{(n)}, \quad n \le \alpha < n+1, n \ge 1$$

1.4 *Integral with Respect To* $(dt)^\alpha$

The next lemma shows the solution of fractional differential equation

$$dy = g(t)(dt)^\alpha, t \ge 0, y(0) = 0, \qquad (1.4.1)$$

by integration with respect to $(dt)^\alpha$.

Lemma. If $g(t)$ is a continuous function, so the solution of (1.4.1) is defined as the following

$$y(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t g(\eta)(d\eta)^\alpha\, d\eta, \quad y(0) = 0$$
$$= \alpha\int_0^t (t-\eta)^{\alpha-1} g(\eta)d\eta, \qquad 0 < \alpha < 1.$$

For more results and various views on fractional calculus, see for example [6, 9, 10, 11, 12, 13, 14, 15].

1.5 *Fractional double Laplace and Sumudu transforms*

Analogous to double Laplace and double Sumudu transform, fractional double Laplace and fractional double Sumudu transforms was established as the following

Definition 1.5.1 [16] Let $f(x,y)$ be where $x,y > 0$, then double Laplace transform of fractional order is given by

$$\mathcal{L}_\alpha^2\{f(x,y)\} = F_\alpha^2(s,p) = \int_0^\infty \int_0^\infty E_\alpha(-(sx + py)^\alpha) f(x,y)\,(dx)^\alpha (dy)^\alpha, \quad (1.5.1)$$

where $s,p \in \mathbb{C}$, and E_α is the Mittag-Leffler function $\sum_{n=0}^{\infty} \frac{x^n}{(\alpha n)!}$.

Definition 1.5.2 [17] Fractional double Sumudu transform of a function $f(x,y)$ is known as

$$S_\alpha^2\{f(x,y)\} = G_\alpha^2(u,v) = \int_0^\infty \int_0^\infty E_\alpha(-(x + y)^\alpha) f(ux,vy)\,(dx)^\alpha (dy)^\alpha, \quad (1.5.2)$$

where $u,v \in \mathbb{C}$, $x,y > 0$ and E_α is the Mittag-Leffler function.

2. Double Natural transform of Fractional Order

In this section, we define the fractional double Natural transform in following way.

Definition 2.1. Let $f(x,t)$ be a function which vanishes for negative values of x and t, then double Natural transform of fractional order (or its fractional double Natural transform) is defined as

$$ {}_\alpha^+\mathbb{N}^2\{f(x,y)\} = {}_\alpha^+\mathcal{R}^2[s,p,(u,v)] = \int_0^\infty \int_0^\infty E_\alpha(-(sx + py)^\alpha) f(ux,vy)\,(dx)^\alpha (dy)^\alpha, \quad (2.1)$$

where $s,p,u,v \in \mathbb{C}$, and E_α is the Mittag-Leffler function.

By using the Mittag-Leffler function property, then the formula (2.1) can be written as follows

$$ {}_\alpha^+\mathbb{N}^2\{f(x,y)\} = {}_\alpha^+\mathcal{R}^2[s,p,(u,v)] = \int_0^\infty \int_0^\infty E_\alpha(-(sx)^\alpha) E_\alpha(-(py)^\alpha) f(ux,vy)\,(dx)^\alpha (dy)^\alpha, \quad (2.2)$$

In particular case, fractional double Natural transform (2.1) turns to double Natural transform as the formula (1.1.3) when $\alpha = 1$.

Remark 2.1 from the above definition we can show that

1. When *u*=*v*=1, we have fractional double Laplace transform, as the formula (1.5.1), which is proposed by [16],
2. When *s*=*p*=1, we have fractional double Sumudu transform as the formula (1.5.2) which is given by [17].

2.1 *Some main properties of the relation between fractional double Natural transform*

Let $f(x,y)$, and $g(x,y)$ be functions whose the fractional double Natural transform exists, then we can get the following general properties of fractional double Natural transform under suitable condition by applying the definition of fractional double Natural transform and fractional calculus as the next

1. Scaling property

$$ {}_\alpha^+\mathbb{N}^2\{f(ax,by)\} = {}_\alpha^+\mathcal{R}^2[s,p,(au,bv)]$$

2. Linearity property is achieved, for any constant *a*, *b* such that

$$ {}_{\alpha}^{+}\mathbb{N}^2\{af(x,y)+bg(x,y)\} = a\,{}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\}+b\,{}_{\alpha}^{+}\mathbb{N}^2\{g(x,y)\} $$

Proof.

1. The result can be get directly by using Definition(2.1)

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(ax,by)\} = \int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)f(aux,bvy)\,(dx)^\alpha(dy)^\alpha = {}_{\alpha}^{+}\mathcal{R}^2[s,p,(au,bv)]. $$

2. We can obtain the result by applying Definition(2.1)

$$ {}_{\alpha}^{+}\mathbb{N}^2\{af(x,y)+bg(x,y)\} = \int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)[af(ux,vy)+bg(ux,vy)]\,(dx)^\alpha(dy)^\alpha, $$

$$ = a\int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)f(ux,vy)\,(dx)^\alpha(dy)^\alpha $$

$$ +b\int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)g(ux,vy)\,(dx)^\alpha(dy)^\alpha $$

$$ =a\,{}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\}+b\,{}_{\alpha}^{+}\mathbb{N}^2\{g(x,y)\} $$

3. The Relationship with Others Transforms

In this section, we will give the relation between fractional double Natural transform and fractional double Laplace transform, further the connection between fractional double Natural transform and fractional double Sumudu transform is obtained.

Theorem 3.1 (fractional double Natural-Laplace duality)

Suppose that fractional double Natural transform of a function $f(x,y)$ exists, and F_α^2 denotes the fractional double Laplace transform, then

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \frac{1}{u^\alpha v^\alpha}F_\alpha^2\left(\frac{s}{u},\frac{p}{v}\right). $$

Proof.

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)f(ux,vy)\,(dx)^\alpha(dy)^\alpha $$

by making the change $\omega = ux$, $\tau = vy$, hence

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \frac{1}{u^\alpha v^\alpha}\int_0^\infty\int_0^\infty E_\alpha\left(-\left(\frac{\omega s}{u}+\frac{\tau p}{v}\right)^\alpha\right)f(\omega,\tau)\,(d\omega)^\alpha(d\tau)^\alpha = \frac{1}{u^\alpha v^\alpha}F_\alpha^2\left(\frac{s}{u},\frac{p}{v}\right). $$

Theorem 3.2 (fractional double Natural-Sumudu duality)

If fractional double Natural transform of a function $f(x,y)$ exists, and G_α^2 denotes the fractional double Sumudu transform, then

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \frac{1}{s^\alpha p^\alpha}G_\alpha^2\left(\frac{u}{s},\frac{v}{p}\right). $$

Proof.

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \int_0^\infty\int_0^\infty E_\alpha(-(sx+py)^\alpha)f(ux,vy)\,(dx)^\alpha(dy)^\alpha $$

by making the change $\omega = sx$, $\tau = py$, hence

$$ {}_{\alpha}^{+}\mathbb{N}^2\{f(x,y)\} = \frac{1}{s^\alpha p^\alpha}\int_0^\infty\int_0^\infty E_\alpha(-(\omega+\tau)^\alpha)f\left(\frac{\omega u}{s},\frac{\tau v}{p}\right)(d\omega)^\alpha(d\tau)^\alpha $$

$$= \frac{1}{s^\alpha p^\alpha} G_\alpha^2\left(\frac{u}{s}, \frac{v}{p}\right)$$

4. The Convolution Theorem of Fractional Double Natural Transform

Theorem 4.1. The convolution of α^{th} order of the functions $f(x,y)$, and $g(x,y)$ can be defined as the follows

$$\left(f(x,y) **_\alpha g(x,y)\right) = \int_0^x \int_0^y f(x-\eta, y-\theta) g(\eta,\theta)\,(d\eta)^\alpha (d\theta)^\alpha,$$

$${}_\alpha^+\mathbb{N}^2\{(f **_\alpha g)(x,y);(u,v,s,p)\} = u^\alpha v^\alpha\, {}_\alpha^+\mathbb{N}^2\{f(x,y);(s,p)\}\, {}_\alpha^+\mathbb{N}^2\{g(x,y);(s,p)\}$$

Proof.
Using the fractional double Natural- Laplace duality from Theorem (3.1), we obtain

$${}_\alpha^+\mathbb{N}^2\{(f **_\alpha g)(x,y);(u,v,s,p)\} = \frac{1}{u^\alpha v^\alpha} F_\alpha^2 (f **_\alpha g)_{\left(\frac{s}{u},\frac{p}{v}\right)}$$

By using the convolution theorem of fractional double Laplace transform

$$\mathcal{L}_\alpha^2\,(f **_\alpha g)_{(x,y)} = \mathcal{L}_\alpha^2\{f(x,y)\}\mathcal{L}_\alpha^2\{g(x,y)\}$$

Then, we have

$${}_\alpha^+\mathbb{N}^2\{(f **_\alpha g)(x,y);(u,v,s,p)\} = \frac{1}{u^\alpha v^\alpha} F_\alpha^2[(f(x,y)]_{\left(\frac{s}{u},\frac{p}{v}\right)} \,.F_\alpha^2[(g(x,y)]_{\left(\frac{s}{u},\frac{p}{v}\right)}$$

$$= u^\alpha v^\alpha\, {}_\alpha^+\mathbb{N}^2\{f(x,y);(s,p)\}\, {}_\alpha^+\mathbb{N}^2\{g(x,y);(s,p)\}.$$

Notation. The above result is in agreement with the convolution of double Natural transform in [3] when $\alpha = 1$.

CONCLUSION

Our work deals with definition of fractional double Natural transform. Some fundamental properties of fractional double Natural transform are obtained and the relation between fractional double Natural - Laplace duality and fractional double Natural-Sumudu duality is given.

Disclaimer

The article has not been previously presented or published, and is not part of a thesis project.

Conflict of Interest

There are no financial, personal, or professional conflicts of interest to declare.

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