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Original article

Mathematical Modeling and Simulation of Spring Constant on the Behavior of Damping in a Mass-spring Damped Harmonic Oscillator

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ABSTRACT

Background and aims. *Mass-Spring systems are second order linear differential equations that have variety of applications in science and engineering. They are the simplest model for mechanical vibration analysis. Damping of the oscillatory system is the effect of preventing or reducing its oscillations gradually with time. The damping ratio in physical systems is produced by the dissipation of stored energy in the oscillation. In this study, the effect of spring constant on the behaviour of damping in a mass-spring harmonic oscillator using simulation was investigated.* ***Methods****. The three cases of damping namely the underdamped, the overdamped and the critically damped cases were studied by varying the spring constant from 2 N/M to 10 N/M. (2, 4, 6, 8 and 10 N/M).* ***Results****. It was found that for the two damping cases; the underdamped case and the overdamped case, the effect of increasing the spring constant on the behavior of harmonic oscillator caused a decrease in the time of damping and an increase in the displacement, while in the case of critically damped harmonic oscillator, there was very little variation in the time of damping or it was almost steady while there was a negative decrease in the displacement.* ***Conclusion****. It was found that for the two cases of underdamped and overdamped harmonic oscillator, the effect of increase in the value of the spring constant causes a decrease in the time of damping, while the effect of increasing the spring constant on the displacement makes it increasing. In the case of critical damped harmonic oscillator, very little variation in the time of damping was noticed with increasing the value of the spring constant. The effect of increasing the spring constant cause a negative decrease in the value of displacement in the critically damped case.*

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INTRODUCTION

Mass-Spring systems are second order linear differential equations that have variety of applications in science and engineering. They are the simplest model for mechanical vibration analysis. Damping occurs when the motion of an oscillator reduces due to an external force. These types of periodic motions of gradually decreasing amplitude are called damped simple harmonic motion. In any damped harmonic oscillator, the energy of the oscillator dissipates or decays continuously with time. The dissipation of energy is mainly due to the frictional forces. The damped system depends on the damping coefficient, which in turn depends on the value of the mass, the value of the spring constant and the value of the damping ratio.

The damping ratio is a measure describing how rapidly the oscillations decay from one bounce to the next. The damping ratio is a dimensionless number and is a system parameter denoted by ξ that can vary from undamped $(\xi = 0)$, underdamped $(\xi < 1)$, through critically damped $(\xi = 1)$ to overdamped $(\xi > 1)$.

The spring constant of a damped harmonic oscillator is an important parameter for characterizing the system's behaviour, and it can be used to calculate other parameters such as the system's natural frequency or the damping coefficient.

Various studies had been carried out by a number of researchers on the mass-spring damping systems. Some workers investigated the effect of mass on the behaviour of the oscillatory system in a mass-spring system. Other researchers studied mathematical modeling and using simulation [1-8]. Gunawan made an analysis of coupled mass-spring damped system by changing spring constant, mass and force [9]. Sunday used the concept of systems theory as an approach to mass-damper-spring and mass-nondamper-spring system [10].

The spring constant k is a measure of the stifness of the spring. It is different for different springs and materials. The larger the spring constant, the stiffer the spring and the more difficult is to stretch. In the case of damped harmonic oscillator, the spring constant of a damped harmonic oscillator is a measure of how much restoring force the spring exerts on the oscillator. It is determined by the spring stiffness, mass and damped coefficient. The aim of this work is to investigate the effect of the spring constant on the behavior of dampers of a simple harmonic oscillator using simulation. The system under study is a mass-spring system.

METHODS

Theory of damped simple harmonic oscillator (S.H.O)

The theory of damped simple harmonic motion is well-understood and is given in details in most textbooks on oscillations. Here, in this paper we will outline the main governing equations in brief. Newton's second law of motion is written as:

$$m\ddot{x} = -sx - r\dot{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where m is the mass, s is the stiffness, r is the constant of proportionality and hass the dimensions of force per unit of velocity { The presence of such a term will always result in energy loss }, and x is the displacement.

The behaviour of the displacement x is obtained from the following equations:

$$m\ddot{x} + r\dot{x} + sx = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where the coefficients m, r and s are constants.

It is seen that when these coefficients are constant, a solution of the form

$$x = C\,e^{\alpha t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Can be found.

There are three possible cases of this solution, each will describe a different behaviour of the displacement x with time. In two of the solutions, the constant C is a constant length, while in the third solution it takes the following form:

$$C = A + Bt^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Where A represents the length, while B is the velocity and t is the time.

Taking C as a constant length leads to the velocity

$$\dot{x} = \alpha\,C\,e^{\alpha t}$$

And the acceleration

$$\ddot{x} = \alpha^{2}\,C\,e^{\alpha t}$$

And equation (2) can be rewritten as

$$C\,e^{\alpha t}\left(m\alpha^{2} + r\alpha + s\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

There are two conditions either, $C\,e^{\alpha t} = 0$ or $\left(m\alpha^{2} + r\alpha + s\right) = 0$

This is a quadratic equation whose solution is given as:

$$\alpha = \frac{-r}{2m} \pm \sqrt{\frac{r^{2}}{4m^{2}} - \frac{s}{m}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

The term $\frac{r^2}{4m^2}$ is the damping resistance and the term $\frac{s}{m}$ is the stifness of the spring.
The displacement can now be expressed as:

$$x_1 = C_1\, e^{\left[\left(-rt/2m\right)+\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{\frac{1}{2}}\right]t},$$

$$x_2 = C_2\, e^{\left[\left(-rt/2m\right)-\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{\frac{1}{2}}\right]t}$$

Or $$x = x_1 + x_2 = C_1\, e^{\left[\left(-rt/2m\right)+\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{\frac{1}{2}}\right]t} + C_2\, e^{\left[\left(\frac{-rt}{2m}\right)-\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{\frac{1}{2}}\right]t} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

Now we come to the most important conditions for the value in the bracket.
The bracket $\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)$ can have a positive value, zero, or can have a negative value depending on the relative magnitude of the two terms inside it.
It must be noted that each of these three conditions gives one of the three possible solutions stated earlier, and each solution describes the behaviour of a particular kind of damping. These conditions are:

1. The value of the bracket may be positive, i.e. $\frac{r^2}{4m^2} > \frac{s}{m}$. Here in this case, the damping resistance term $\frac{r^2}{4m^2}$ dominates the stifness term $\frac{s}{m}$ which results in a heavy damping and causes a dead beat oscillating system.
2. The value of the bracket is zero, i.e. $\frac{r^2}{4m^2} = \frac{s}{m}$, which is the case where the damping resistance term equals to the stifness term. So the balance of these two terms results in a critically damped oscillating system.
3. The value of the bracket $\frac{r^2}{4m^2} < \frac{s}{m}$ is negative, which means that the stifness term dominates the damping resistance term. Here, in this case the system is lightly damped and gives oscillatory damped simple harmonic motion.

Case1: Heavy damping

By letting $\frac{r}{2m} = p$ and $\left(\frac{r^2}{4m^2}-\frac{s}{m}\right)^{\frac{1}{2}} = q$ we can write equation (7) as

$$x = e^{-pt}\left(C_1 e^{qt} + C_2 e^{-qt}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Where C_1 and C_2 are arbitrary in value but have the same dimensions as C. We must note here that the two separate values of C are allowed because the differential equation (1) is second order.
Now if we, and put $C_1 + C_2 = F$, and $C_1 - C_2 = G$, the equation for the displacement takes the form:

$$x = e^{-pt}\left[\frac{F}{2}\left(e^{qt}+e^{-qt}\right)+\frac{G}{2}\left(e^{qt}-e^{-qt}\right)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Or

$$x = e^{-pt}\left(F\cosh qt + G\sinh qt\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Equation (10) represents a non-oscillatory behaviour, but the actual displacement will depend upon the initial conditions that is, the value of x at time $t = 0$.
If $x = 0$ at time $t = 0$, then $F = 0$ and we get the displacement

$$x = G\,e^{\frac{-rt}{2m}} \sinh\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{\frac{1}{2}} t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Case2: Critical damping $\left(\frac{r^2}{4m^2} = \frac{s}{m}\right)$

By using the same notation of case1, we can see that $q = 0$ and that the displacement takes the form:

$$x = e^{-pt}(C_1 + C_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

This represents the limiting case of the behaviour of case1 as the value of q changes from positive to negative. In this case the quadratic equation in α has equal roots, which in a differential equation solution demands that the constant C must be written as $C = A + Bt$ where A is a constant length and B is a given velocity which depends on the initial or boundary conditions and it can be verified that the value

$$x = (A + Bt)e^{\frac{-rt}{2m}} = (A + Bt)e^{-pt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Satisfies the equation $m\ddot{x} + r\dot{x} + sx = 0$ where $\frac{r^2}{4m^2} = \frac{s}{m}$.

Case3: Damped S.H.M.

As stated before, when $\frac{r^2}{4m^2} < \frac{s}{m}$, the damping becomes light, and this case is the most important kind of behaviour which is the oscillatory damped simple harmonic motion.

The expression $\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{\frac{1}{2}}$ is an imaginary quantity, because the number under the square root is a negative number. This can be rewritten as:

$$\pm\left(\frac{r^2}{4m^2} - \frac{s}{m}\right)^{\frac{1}{2}} = \pm\sqrt{-1}\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)^{\frac{1}{2}} = \pm i\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)^{\frac{1}{2}} \text{ where } i = \sqrt{-1}$$

So the displacement is given by:

$$x = C_1 e^{\frac{-rt}{2m}}\, e^{+i\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)^{\frac{1}{2}} t} + C_2 e^{\frac{-rt}{2m}}\, e^{-i\left(\frac{s}{m} - \frac{r^2}{4m^2}\right)^{\frac{1}{2}} t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

RESULTS AND DISCUSSION

Figure 1(a-e) shows the effect of changing the spring constant for values of spring constants 2 ,4, 6, 8 and 10 N/M while keeping the mass of the spring at 1 kg and c=2, for the underdamped harmonic oscillator.

From these figures, it can be seen that the time of damping decreases with increasing the value of k, e.g., for k=2 N/M, t=4 sec, and for k= 10 N/M, t=1.1 sec. It is also noticed that the displacement increases from $-2\times10^{4}\ m$ to about $3.8\times10^{-4}\ m$ with the increase in the value of the spring constant.

The effect of changing the spring constant on the behaviour of damping for the case of overdamped harmonic oscillator is shown in figure 2(a-e). It can be seen from these figures that as the value of the spring constant increases the time of damping decreases. The same picture also holds for the effect of spring constant on displacement as in figure 1, in which the displacement increased with increasing the spring constant but more slowly.

Figures 3(a-e) presents the effect of the spring constant of the behaviour of damping for the case of critical damped harmonic oscillator. It is noticed in this case that there was a very little variation in the time of damping or we can say that it is nearly steady with the increase of the spring constant, while the effect on the displacement was clear, as the displacement decreased in this case from a negative value of $-1.1\times10^{-4} m \quad to \quad -0.2\times10^{-4} m$.

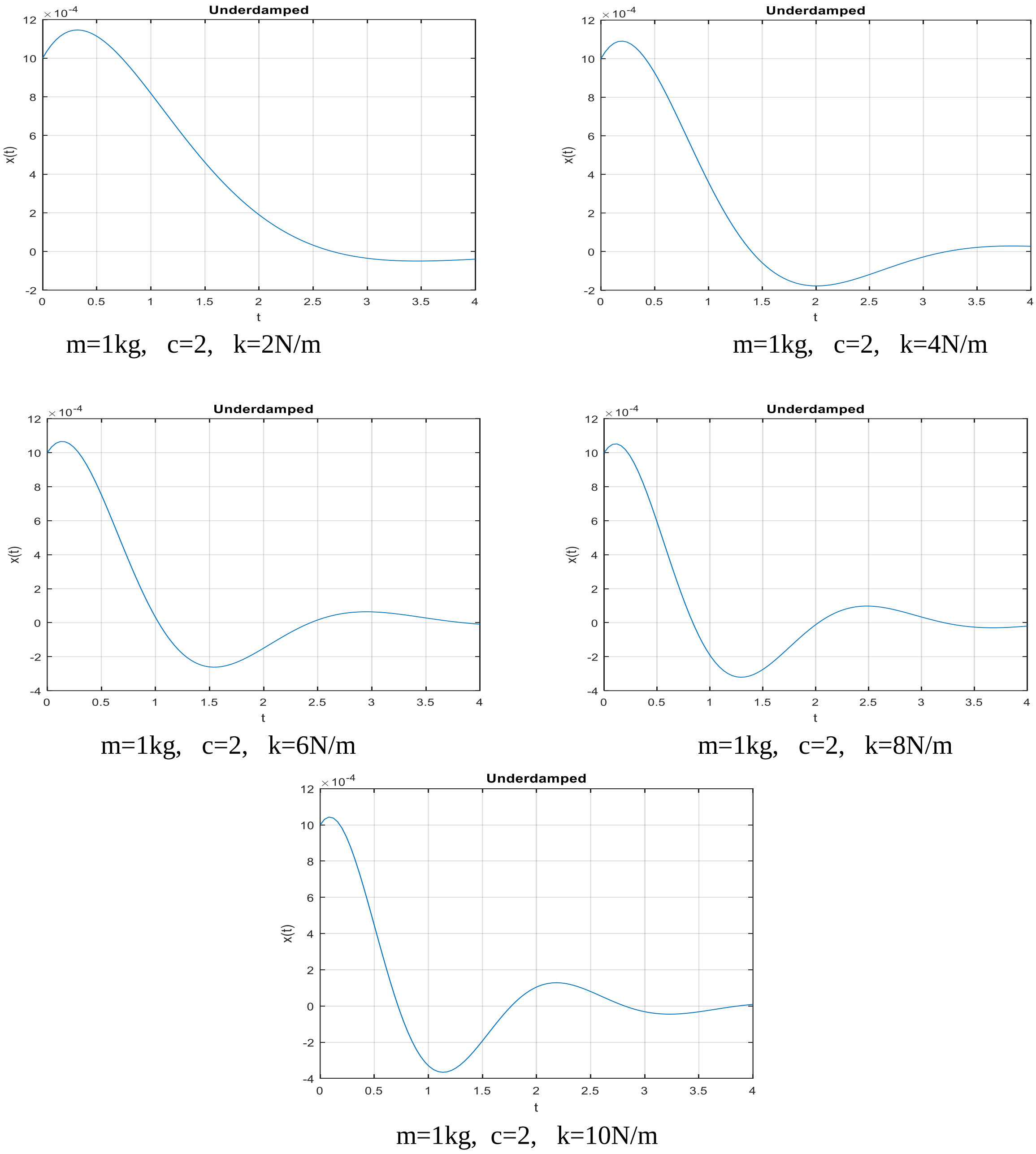
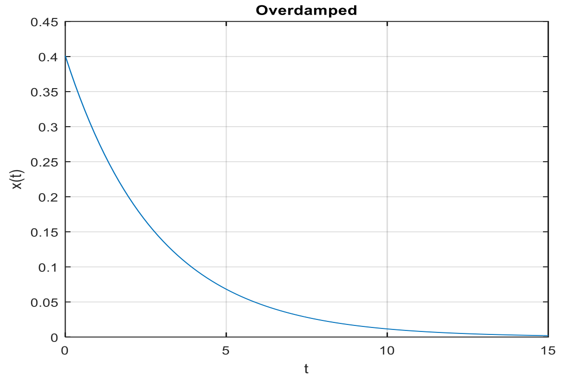
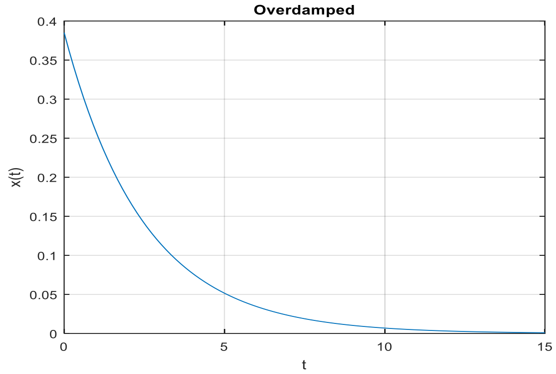


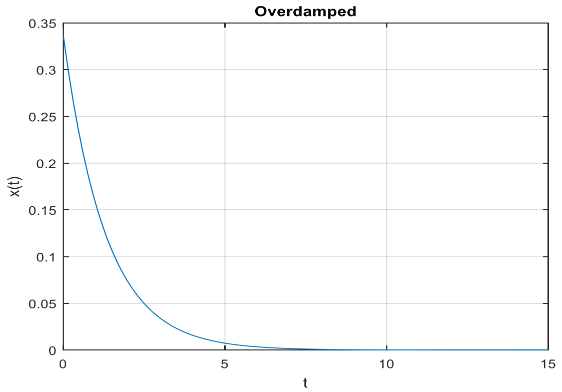
Figure 1. The effect of changing the spring constant on the behaviour of harmonic oscillator for the underdamped case.



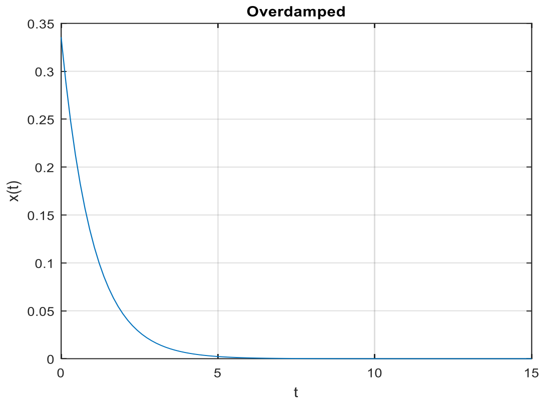
M=100kg, c=600, k=200N/m



M=100kg, c=600, k=225N/m

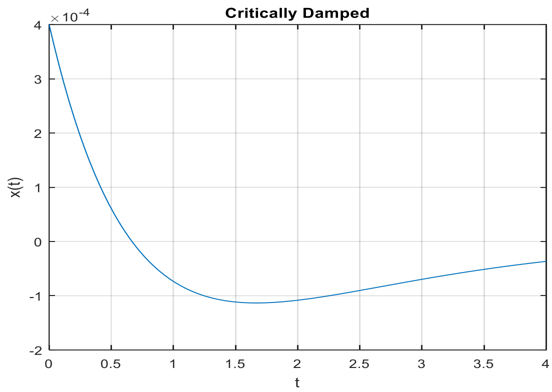


M=100kg, c=600, k=400N/m

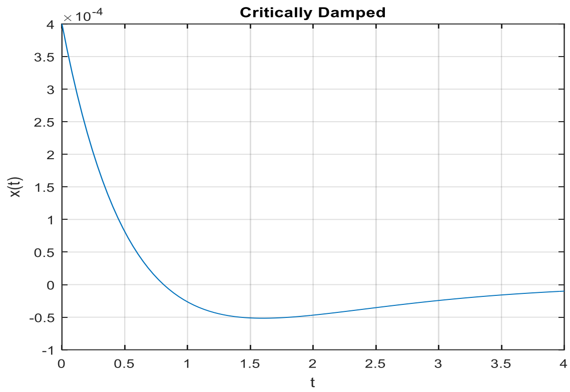


M=100kg, c=600, k=500N/m

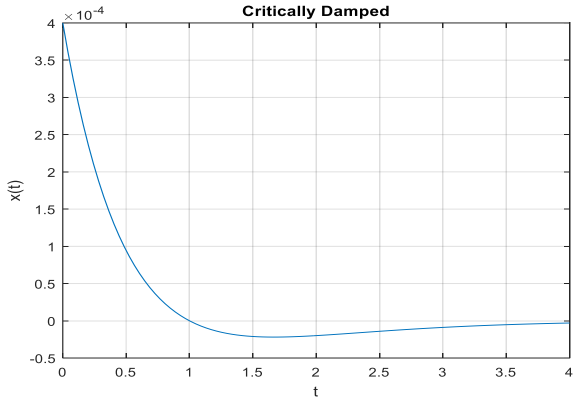
Figure 2. The effect of changing the spring constant on the behaviour of harmonic oscillator for the overdamped case.



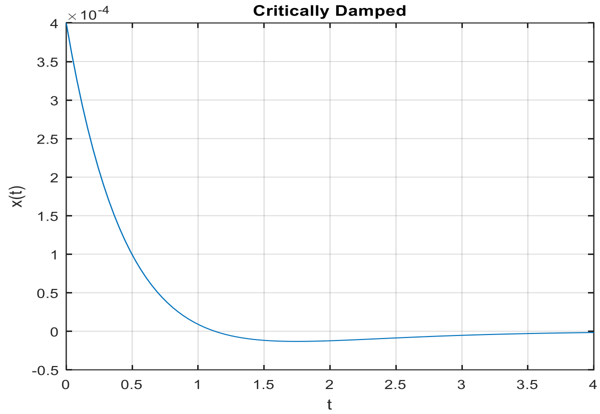
M=100kg,K=100N/M, C=300



M=100kg,K=160N/M, C=300



M=100kg, K=225N/M, C=300



M=100kg, K=260N/M, C=300

Figure 3. The effect of changing the spring constant on the behaviour of harmonic oscillator for the critically damped case.

CONCLUSION

From this study, it was found that for the two cases of underdamped and overdamped harmonic oscillator, the effect of changing the spring constant causes a decrease in the time of damping with the increase in the value of the spring constant, while the effect of increasing the spring constant on the displacement makes it increasing. In the case of critical damped harmonic oscillator, very little variation in the time of damping was noticed with increasing the value of the spring constant. The effect of increasing the spring constant cause a negative decrease in the value of displacement in the critically damped case.

Conflict of Interest

There are no financial, personal, or professional conflicts of interest to declare.

REFERENCES

1. Al-Nuaimi A, Ola A. Mohammed. Study the Effects of Mass in a Simple Damped Harmonic Motion of a Mass-spring System Using Simulation, Libyan Journal of Basic Science, 2021;14(1):1-11.
2. Gomez J, Rosales J, Bernal J. Mathematical modeling of the mass-spring damper system-A fractional calculus approach, Acta Universitaria. 2012;22(5):5-11.
3. Unal O, Akkas N. Numerical simulation of the mass-spring-damper system using high resolution schemes. Journal of Marine and Engineering Technology. 2022;2(2):58-67.
4. Rusianto T. Mathematical Simulation of damping oscillation on motorcycle suspension. Inter J Diff Equ. 2023;18(1):49-57.

5. Shiko S. Simulation of a spring mass damper system using MATLAB, Thesis, publication of the department of mechanical engineering, University of lagos, Akoka, Yab, Lagos, Nigeria, (2009).
6. Nandini C. Simulation of simple harmonic oscillator using Xcos. JETIR. 2017;4(9):831-839.
7. Ingale A, Daga A, Naik R. Modeling mass-spring damper system using simscape. International Journal of Engineering Research and Application. 2018;8(1):30-33.
8. Melanie, Garcia. GUI Matlab code to display damped, undamped, forced and unforced mass spring system. Thesis, University of Arkansas, Fayetteville (2015).
9. Michelle S Gunawan. Analysis of coupled mass-spring-damper system by changing spring constant, mass, and force. Proceedings of annual conference on management and information technology, 2017. Master of information technology department, Faculty of engineering and information technology, SGU.
10. Sunday S. Systems Theory: An approach to mass-damper-spring and mass-nondamper-spring systems. General Letters in Mathematic. 2017;3(3):169-176.

النمذجة والمحاكاة الرياضية لثابت الزنبرك على سلوك التخميد في مذبذب توافقي مخفف بكتلة الربيع

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المستخلص

الخلفية والأهداف. أنظمة Mass-Spring هي معادلات تفاضلية خطية من الدرجة الثانية لها تطبيقات متنوعة في العلوم والهندسة. إنها أبسط نموذج لتحليل الاهتزاز الميكانيكي. التخميد في النظام التذبذب هو تأثير منع أو تقليل اهتزازاته تدريجياً مع مرور الوقت. يتم إنتاج نسبة التخميد في الأنظمة الفيزيائية عن طريق تبديد الطاقة المخزنة في التذبذب. في هذه الدراسة تم دراسة تأثير ثابت الزنبرك على سلوك التثبيط في المذبذب التوافقي الكتلي النابض باستخدام المحاكاة. **طُرق الدراسة**. تمت دراسة حالات التخميد الثلاث وهي الحالات المنخفضة التخميد والحالات المثبطة للغاية والمثبطة بشكل خطير من خلال تغيير ثابت الزنبرك من 2 N / M إلى 10 N / M. (2 و 4 و 6 و 8 و 10 N / M). **النتائج**. وجد أن في حالتي التخميد ؛ الحالة المنخفضة التخميد والحالة المفرطة التخميد ، أدى تأثير زيادة ثابت الزنبرك على سلوك المذبذب التوافقي إلى انخفاض وقت التخميد وزيادة الإزاحة ، بينما في حالة المذبذب التوافقي المثبط بشكل حاسم ، كان هناك القليل جدًا الاختلاف في وقت التخميد أو كان ثابتًا تقريبًا بينما كان هناك انخفاض سلبي في الإزاحة. **الخاتمة**. يؤدي تأثير زيادة ثابت الزنبرك إلى انخفاض سلبي في قيمة الإزاحة في الحالة المثبطة بشكل خطير.

الكلمات الدالة. نظام كتلة الربيع ، مذبذب توافقي مخفف ، ثابت الربيع ، النمذجة ، المحاكاة.