

# Spectral Analysis of Non-Self-Adjoint Operators and Hopf Bifurcation in Cardiac Reaction-Diffusion Systems

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## Abstract

Myocardial infarction and regional myocardial ischemia are associated with complex and variable clinical manifestations and profound cellular and tissue-level alterations. At the cellular level, ischemia causes metabolic stress, hypoxia and ionic imbalance. These changes can affect membrane excitability, ion-channel function and impulse conduction, contributing to abnormal impulse generation, abnormal automaticity, triggered activity and reentrant mechanisms associated with ischemia-related cardiac arrhythmias [1, 5, 17]. From a dynamical-systems perspective, some of these electrophysiological alterations can be represented by periodic solutions and, under appropriate dynamical and stability conditions, limit cycles. The emergence of such oscillatory behavior can be investigated mathematically through Hopf bifurcation, while distinguishing the mathematical representation of oscillations from the underlying physiological mechanisms [3, 4, 8, 12]. This study employs functional analytic methods to analyze coupled nonlinear reaction-diffusion systems governing membrane potential propagation in cardiac tissue. The model is formulated as an abstract semilinear equation on the complex Hilbert space  $X = L^2(\Omega) \times L^2(\Omega)$ . The analysis relies on continuous analytic  $C_0$ -semigroups and the Banach fixed point theorem. We further study the spectral properties of the linearized non-self-adjoint differential operator around an equilibrium state, with spatially varying diffusion  $D(x)$  and ionic currents. Under the spectral transversality condition, we prove the occurrence of an infinite-dimensional Hopf bifurcation at the critical parameter value  $\mu_c$ . For a one-dimensional tissue strip  $\Omega = (0, \pi)$ , the analytical and numerical results show that structural and electrophysiological heterogeneities can cause eigenvalues of the linearized operator to cross the imaginary axis. In particular, the fundamental spatial mode  $k = 0$  crosses into the right half-plane while higher spatial modes remain spectrally stable under the parameter system considered. This spectral transition provides a mathematical representation of the emergence of sustained oscillatory dynamics and may be relevant to the study of abnormal electrical activity in ischemic tissue.

**Keywords.** Reaction-Diffusion Systems, Hopf Bifurcation, Spectral Analysis, Limit Cycles, Cardiac Arrhythmias.

## Introduction

Cardiac electrophysiology provides an important application of mathematical analysis to the study of electrical activity in cardiac tissue. Mathematical models of excitable tissue have evolved since the work of Hodgkin and Huxley [9], followed by the simplified membrane models of Fitzhugh [7] and Nagumo et al. [14]. From a functional analytic perspective, Henry [8] developed the geometric theory for semilinear parabolic equations using analytic  $C_0$ -semigroups. Crandall and Rabinowitz [3] studied bifurcation from simple eigenvalues and the Hopf bifurcation theorem in infinite dimensions [4]. These results are relevant to the study of periodic solutions arising from an equilibrium when a pair of complex conjugate eigenvalues crosses the imaginary axis under the spectral transversality condition. The general theory of bifurcation and perturbation theory for linear operators also provides the tools for studying parameter-dependent operators [2, 10, 12]. In mathematical biology, Keener and Sneyd [11] and Murray [13] studied electrical wave propagation in living tissues using reaction-diffusion systems.

The spectral analysis of non-self-adjoint differential operators remains challenging when the diffusion coefficient  $D(x)$  and ionic parameters vary spatially. The studies cited above address different aspects of the present problem. Henry [8] developed the analytical theory of semilinear parabolic equations, while Crandall and Rabinowitz [3, 4] studied bifurcation from simple eigenvalues and the Hopf theorem in infinite dimensions. Operator perturbation theory by Kato [10] and bifurcation theory by Chow and Hale [2] and Kielhöfer [12] offer tools for studying parameter-dependent spectra. In mathematical physiology, Keener and Sneyd [11] and Murray [13] examined reaction-diffusion models of electrical activity in excitable tissue. Although these studies provide the mathematical and physiological background for the present work, they do not establish the spectral conclusions for the specific non-self-adjoint operator considered here.

Classical literature on infinite-dimensional Hopf bifurcation, such as the works of Henry, Crandall, and Rabinowitz, focuses on abstract operator settings. These works do not address the specific structure of non-self-adjoint Jacobian operators ( $\mathcal{L}_\mu \neq \mathcal{L}_\mu^*$ ) arising from reaction-diffusion systems in cardiac electrophysiology. This work extends those abstract results to account directly for the interplay between membrane potential and ionic dynamics. Theorems 2.2 and 2.3 derive the spectral conditions using Fourier mode decomposition. This analysis shows that instability and temporal oscillations are restricted to the spatial mode ( $k = 0$ ), while spatial diffusion guarantees strict linear stability across all higher modes ( $k \geq 1$ ), confirming the spectral gap. The study also derives an analytical expression for the crossing speed  $\alpha'(\mu_c) = -\frac{\varepsilon}{2} \neq 0$  in terms of the model's physiological parameters by connecting Lemma 1.3 with Theorem 2.3. This study aims to analyze the spectrum of the linearized non-self-adjoint operator associated with the coupled cardiac reaction-diffusion system and to examine how its eigenvalues depend on the parameter  $\mu$ . The main contribution is to determine the spectral conditions under which a simple conjugate pair of eigenvalues crosses the imaginary axis and to establish the resulting infinite-

dimensional Hopf bifurcation, subject to the required hypotheses. The analysis also considers the one-dimensional domain  $\Omega = (0, \pi)$  to examine the spectral behavior of the system for the parameter range studied. This study addresses the following research questions:

- (i) What assumptions guarantee the existence and uniqueness of a mild solution to the coupled reaction-diffusion system?
- (ii) How does the spectrum of the linearized non-self-adjoint operator  $\mathcal{L}_\mu$  change as the real parameter  $\mu$  varies?
- (iii) Which spectral conditions ensure that a simple conjugate pair of eigenvalues  $\lambda_+(\mu)$  and  $\lambda_-(\mu)$  crossing the imaginary axis leads to an infinite-dimensional Hopf bifurcation?
- (iv) What do the analytical and numerical results for the one-dimensional  $\Omega = (0, \pi)$  reveal about the spectral transition and the existence of local periodic solutions?

The following definitions and preliminary results are needed for the analysis.

### Definition 1.1[8, 15]

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with a sufficiently smooth boundary and let

$$X = L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$$

be the associated complex Hilbert space equipped with its standard inner product. Consider the abstract semilinear evolution equation

$$\frac{dU}{dt} = \mathcal{A}U + \mathcal{F}(U), \quad U(0) = U_0 \in X,$$

where,  $\mathcal{A}: D(\mathcal{A}) \subset X \rightarrow X$  is a linear, generally unbounded operator that generates analytic  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  and  $\mathcal{F}: X \rightarrow X$  is a nonlinear operator satisfying a local Lipschitz condition. A continuous function  $U \in C([0, \tau], X)$  is called a mild solution of the above problem if, for every  $t \in [0, \tau]$ ,

$$U(t) = T(t)U_0 + \int_0^t T(t-s)\mathcal{F}(U(s))ds.$$

This integral formulation is used when a classical derivative at the initial time is not required.

### Definition 1.2[6, 10]

Let  $\mathcal{A}_\mu: D(\mathcal{A}_\mu) \subset X \rightarrow X$  be a closed, densely defined linear differential operator depending on a real parameter  $\mu \in \mathbb{R}$ .

The operator  $\mathcal{A}_\mu$  is called non-self-adjoint if  $\mathcal{A}_\mu \neq \mathcal{A}_\mu^*$ , where  $\mathcal{A}_\mu^*$  denotes its adjoint operator.

The spectrum of  $\mathcal{A}_\mu$  is defined by

$$\sigma(\mathcal{A}_\mu) = \{\lambda \in \mathbb{C}: \lambda I - \mathcal{A}_\mu \text{ is not boundedly invertible}\}.$$

The point spectrum is the subset

$$\sigma_p(\mathcal{A}_\mu) = \{\lambda \in \sigma(\mathcal{A}_\mu): \exists \phi \in D(\mathcal{A}_\mu), \phi \neq 0, \mathcal{A}_\mu \phi = \lambda \phi\}.$$

If  $\mathcal{A}_\mu$  has a compact resolvent, its spectrum consists of isolated eigenvalues of finite algebraic multiplicity, with possible accumulation only at infinity.

### Definition 1.3[4,8,12]

Let  $U_\mu^* \in D(\mathcal{A}_\mu)$  denote an equilibrium of the parameter-dependent system, satisfying

$$\mathcal{A}_\mu U_\mu^* + \mathcal{F}(U_\mu^*) = 0.$$

Assume that  $\mathcal{F}$  is Fréchet differentiable in a neighborhood of  $U_\mu^*$  and that  $D\mathcal{F}(U_\mu^*)$  is bounded on  $X$ . The linearized operator about the equilibrium is defined by

$$\mathcal{L}_\mu = \mathcal{A}_\mu + D\mathcal{F}(U_\mu^*),$$

with domain

$$D(\mathcal{L}_\mu) = D(\mathcal{A}_\mu).$$

Suppose that  $\mathcal{L}_\mu$  possesses a pair of isolated simple complex conjugate eigenvalues  $\lambda(\mu) = \alpha(\mu) \pm i\omega(\mu)$ , where  $\alpha(\mu), \omega(\mu) \in \mathbb{R}$ .

We say that the system satisfies the spectral transversality condition at the critical parameter value  $\mu = \mu_c$  if

$$\lambda(\mu_c) = \pm i\omega_0, \quad \omega_0 > 0,$$

and

$$\left. \frac{d}{d\mu} \operatorname{Re} \lambda_+(\mu) \right|_{\mu=\mu_c} = \alpha'(\mu_c) \neq 0.$$

In addition, for the Hopf bifurcation setting, we require that no other spectral value of  $\mathcal{L}_{\mu_c}$  lies on the imaginary axis. Thus,  $\sigma(\mathcal{L}_{\mu_c}) \cap i\mathbb{R} = \{i\omega_0, -i\omega_0\}$ . The nonzero derivative condition guarantees that the conjugate pair crosses the imaginary axis with nonzero speed as the parameter varies. This is the essential spectral crossing mechanism underlying the Hopf analysis in infinite-dimensional settings.

### Definition 1.4 [8, 12]

When the spectral conditions in Definition 1.3 are satisfied and the hypotheses of the infinite-dimensional Hopf bifurcation theorem hold, a local branch of nontrivial periodic solutions bifurcates from the equilibrium at  $\mu = \mu_c$ . The periods  $T_\mu$  satisfy

$$\lim_{\mu \rightarrow \mu_c} T_\mu = \frac{2\pi}{\omega_0}$$

The existence of periodic solutions does not determine their stability by itself. The following preliminary results provide the necessary results for the existence, spectral, and bifurcation analysis presented below. Specifically, they cover the existence of mild solutions, the spectral structure of the linearized operator and the parameter dependence of simple isolated eigenvalues.

**Lemma 1.1**[8, 15]

Let  $\{T(t)\}_{t \geq 0}$  be a strongly continuous analytic semigroup generated by  $A$  on the Hilbert space  $X$ , satisfying

$$\|T(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}, \quad t \geq 0,$$

for some constants  $M \geq 1$  and  $\omega \in \mathbb{R}$ .

Suppose that the nonlinear operator

$$\mathcal{F}: X \rightarrow X$$

satisfies the local Lipschitz condition

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_X \leq L(R)\|U - V\|_X,$$

for all  $U, V \in B_R(0)$ , where  $L(R) > 0$  denotes a Lipschitz constant depending on  $R$ . Then, for every initial value  $U_0 \in X$ , there exists a time

$$T_0 = T_0(U_0, R) > 0$$

such that the abstract Cauchy problem admits a unique local mild solution

$$U \in C([0, T_0]; X).$$

Moreover, the solution is represented by

$$U(t) = T(t)U_0 + \int_0^t T(t-s)\mathcal{F}(U(s))ds.$$

The result follows from the contraction-mapping argument applied to the corresponding integral operator [8, 15].

**Lemma 1.2**

Suppose that  $\Omega \subset \mathbb{R}^n$  is a bounded domain with sufficiently smooth boundary and that the diffusion coefficient satisfies the uniform ellipticity condition

$$D(x) \geq d_0 > 0, \quad x \in \overline{\Omega}.$$

Assume further that the coefficients of the linearized operator  $\mathcal{L}_\mu$  possess the regularity required for the elliptic realisation of the operator and that the imposed boundary conditions yield a compact embedding

$$D(\mathcal{L}_\mu) \Subset X.$$

Then  $\mathcal{L}_\mu$  has a compact resolvent. Consequently, its spectrum consists entirely of isolated eigenvalues of finite algebraic multiplicity:

$$\sigma(\mathcal{L}_\mu) = \sigma_p(\mathcal{L}_\mu),$$

and the only possible accumulation point of the eigenvalues is at infinity. In particular, the spectral analysis of the linearized cardiac reaction-diffusion system can be reduced to the study of its eigenvalues and their dependence on the physiological parameter  $\mu$ .

**Lemma 1.3**

Let  $\lambda(\mu)$  be a simple isolated eigenvalue of the operator  $\mathcal{L}_\mu$ , with corresponding eigenvector

$$\phi(\mu) \in D(\mathcal{L}_\mu), \quad \phi(\mu) \neq 0.$$

Let  $\phi^*(\mu) \in D(\mathcal{L}_\mu^*)$  be the corresponding eigenvector of the adjoint operator satisfying the normalization condition

$$\langle \phi(\mu), \phi^*(\mu) \rangle_X = 1.$$

Suppose that the operators  $\mathcal{L}_\mu$  have a common domain and that the mapping

$$\mu \mapsto \mathcal{L}_\mu$$

is continuously differentiable in the appropriate operator topology.

Then the simple isolated eigenvalue  $\lambda(\mu)$  is differentiable with respect to  $\mu$ , and

$$\left. \frac{d\lambda}{d\mu} \right|_{\mu=\mu_c} = \left\langle \left. \frac{d\mathcal{L}_\mu}{d\mu} \right|_{\mu=\mu_c} \phi(\mu_c), \phi^*(\mu_c) \right\rangle_X.$$

Consequently, if  $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ , then

$$\alpha'(\mu_c) = \operatorname{Re} \left[ \left. \frac{d\lambda}{d\mu} \right|_{\mu=\mu_c} \right].$$

Hence, the spectral transversality condition is satisfied whenever

$$\operatorname{Re} \left\langle \left. \frac{d\mathcal{L}_\mu}{d\mu} \right|_{\mu=\mu_c} \phi(\mu_c), \phi^*(\mu_c) \right\rangle_X \neq 0$$

which provides a direct analytical criterion for establishing the nonzero crossing speed required in the Hopf bifurcation analysis [10].

## Results

This section presents the main theoretical results of the study, linking the spectral properties of non-self-adjoint differential operators to the nonlinear dynamics of ischemic cardiac tissue. We first establish global-in-time existence and uniqueness of mild solutions for the semilinear reaction-diffusion system under appropriate dissipativity assumptions. Next, we characterize the spectral structure of the linearized operator to identify the instability mechanism and under the hypotheses of an infinite-dimensional Hopf bifurcation theorem, we establish the existence of a local branch of nontrivial periodic solutions at a critical parameter value.

### Theorem 2.1

Suppose that the nonlinear operator

$$\mathcal{F}: X \rightarrow X, \quad X = L^2(\Omega) \times L^2(\Omega), \quad \|U\|_X = \sqrt{\langle U, U \rangle_X}.$$

Consider the semilinear evolution problem

$$\frac{dU}{dt} = \mathcal{A}U + \mathcal{F}(U), \quad U(0) = U_0 \in X,$$

and there exist constants  $\omega_1 > 0$ ,  $k_1 > 0$ ,  $k_2 \geq 0$ , such that

$$\begin{aligned} \operatorname{Re}\langle \mathcal{A}U, U \rangle_X &\leq -\omega_1 \|U\|_X^2, \\ \operatorname{Re}\langle \mathcal{F}(U), U \rangle_X &\leq -k_1 \|U\|_X^2 + k_2. \end{aligned}$$

Then the abstract Cauchy problem has a unique mild solution

$$U \in C([0, \infty); X).$$

Moreover, the solution satisfies the a priori estimate

$$\|U(t)\|_X^2 \leq \|U_0\|_X^2 e^{-2c_1 t} + \frac{c_2}{c_1} (1 - e^{-2c_1 t}), \quad t \geq 0,$$

$$c_1 = \omega_1 + k_1 > 0, \quad c_2 = k_2.$$

Consequently, the solution remains uniformly bounded on every finite time interval and can be continued for all  $t \geq 0$ .

Under the corresponding regularity assumptions on  $\mathcal{F}$  and the analytic semigroup, the solution is classical for  $t > 0$ , with

$$U \in C((0, \infty); D(\mathcal{A})) \cap C^1((0, \infty); X),$$

For the definitions and properties of these functional spaces and operators  $\mathcal{A}$ ,  $\mathcal{C}$  and  $\mathcal{C}^1$ , see the standard references [8, 15].

### Proof

By Lemma 1.1, the local Lipschitz continuity of  $\mathcal{F}$  and the analytic  $\mathcal{C}_0$  continuity of  $\mathcal{F}$  and the analytic  $\mathcal{C}_0$ -semigroup generated by  $\mathcal{A}$  imply the existence of a unique local mild solution satisfying the variation-of-constants formula

$$U(t) = T(t)U_0 + \int_0^t T(t-s)\mathcal{F}(U(s))ds.$$

Thus, there exists  $T_0 > 0$ , depending on the initial data and the local bounds for  $\mathcal{F}$ , such that

$$U \in C([0, T_0]; X).$$

It remains to show that the solution cannot blow up in finite time. Taking the real part of the inner product of the evolution equation with  $U(t)$ , we obtain

$$\frac{1}{2} \frac{d}{dt} \|U(t)\|_X^2 = \operatorname{Re}\langle \mathcal{A}U(t), U(t) \rangle_X + \operatorname{Re}\langle \mathcal{F}(U(t)), U(t) \rangle_X.$$

Using the dissipativity assumptions gives

$$\frac{d}{dt} \|U(t)\|_X^2 \leq -\omega_1 \|U(t)\|_X^2 - k_1 \|U(t)\|_X^2 + k_2.$$

Hence,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U(t)\|_X^2 &\leq -c_1 \|U(t)\|_X^2 + c_2, \\ c_1 &= \omega_1 + k_1 > 0. \end{aligned}$$

Equivalently,

$$\frac{d}{dt} \|U(t)\|_X^2 \leq -2c_1 \|U(t)\|_X^2 + 2c_2.$$

Gronwall's inequality therefore yields

$$\|U(t)\|_X^2 \leq \|U_0\|_X^2 e^{-2c_1 t} + \frac{c_2}{c_1} (1 - e^{-2c_1 t}).$$

In particular,

$$\sup_{t \geq 0} \|U(t)\|_X^2 \leq \max \left\{ \|U_0\|_X^2, \frac{c_2}{c_1} \right\} < \infty.$$

Thus, the local solution remains bounded and can be extended successively to arbitrary finite intervals. Hence it exists uniquely for every  $t \geq 0$ , proving global-in-time existence and uniqueness. The stated regularity for  $t > 0$  follows under the corresponding smoothing and regularity assumptions for the analytic semigroup.

**Theorem 2.2**

Let

$$\mathcal{A}_\mu: D(\mathcal{A}_\mu) \subset X \rightarrow X.$$

Assume that  $\mathcal{A}_\mu$  has compact resolvent and depends continuously, and sufficiently smoothly when required, on  $\mu$ . Suppose that there exists a critical parameter value  $\mu_c$  such that:

1. At  $\mu = \mu_c$ , the spectrum contains a simple conjugate pair

$$\lambda(\mu_c) = \pm i\omega_0, \quad \omega_0 > 0.$$

2. These are the only spectral points on the imaginary axis:

$$\sigma(\mathcal{A}_{\mu_c}) \cap i\mathbb{R} = \{i\omega_0, -i\omega_0\}.$$

3. All remaining spectral values satisfy the spectral gap condition

$$\operatorname{Re} \lambda \leq -\delta < 0$$

4. The critical eigenvalues depend on  $\mu$  in a neighborhood of  $\mu_c$ .

Then the spectrum of  $\mathcal{A}_\mu$  has the spectral configuration required for an infinite-dimensional Hopf bifurcation, provided the transversality condition

$$\frac{d}{d\mu} \operatorname{Re} \lambda_+(\mu) \Big|_{\mu=\mu_c} \neq 0.$$

**Proof**

By Lemma 1.2, under the ellipticity, domain, boundary-condition and coefficient assumptions imposed on the reaction-diffusion operator, the resolvent operator  $(\lambda I - \mathcal{A}_\mu)^{-1}$  has a compact resolvent for  $\lambda \in \rho(\mathcal{A}_\mu)$ , where  $\rho(\mathcal{A}_\mu) = \mathbb{C} \setminus \sigma(\mathcal{A}_\mu)$  represents the resolvent set of the operator. Consequently, the spectrum  $\sigma(\mathcal{A}_\mu)$  consists of isolated eigenvalues of finite algebraic multiplicity, with no finite accumulation point. Under the assumed regular dependence of  $\mathcal{A}_\mu$  on  $\mu$ , the isolated eigenvalues vary continuously when the corresponding perturbation hypotheses are satisfied. Because  $\mathcal{A}_\mu$  is non-self-adjoint, its spectrum is not restricted to the real axis. Since the coefficients and boundary conditions of the present model are real, complex eigenvalues occur in conjugate pairs. Thus, when

$$\lambda_+(\mu_c) = i\omega_0 \text{ and } \lambda_-(\mu_c) = \overline{\lambda_+(\mu_c)} = -i\omega_0.$$

The supposed spectral gap implies that every other eigenvalue remains strictly in the left half-plane:

$$\operatorname{Re} \lambda_k(\mu_c) \leq -\delta < 0, \quad k \geq 1.$$

Therefore,

$$\sigma(\mathcal{A}_{\mu_c}) \cap i\mathbb{R} = \{\pm i\omega_0\},$$

**Theorem 2.3**

Suppose that the hypotheses of Theorems 2.1 and 2.2 hold. In addition, assume that:

1.  $\pm i\omega_0$  are simple isolated eigenvalues of  $\mathcal{A}_{\mu_c}$ ;
2. no other spectral value lies on the imaginary axis;
3. the nonlinear term has the required smoothness near the equilibrium;
4. the spectral transversality condition holds:

$$\alpha'(\mu_c) = \frac{d}{d\mu} \operatorname{Re} \lambda(\mu) \Big|_{\mu=\mu_c} \neq 0.$$

Then the abstract semilinear evolution equation undergoes a local infinite-dimensional Hopf bifurcation at  $\mu = \mu_c$ . In particular, there exists a local branch of nontrivial periodic solutions  $U_\mu(t)$  where,

$$T(\mu) \rightarrow \frac{2\pi}{\omega_0} \text{ as } \mu \rightarrow \mu_c.$$

**Proof**

Let  $\phi_0 \in D(\mathcal{A}_{\mu_c})$  and  $\lambda(\mu_c) = i\omega_0$ .  $\phi_0^* \in D(\mathcal{A}_{\mu_c}^*)$  such that  $\langle \phi_0, \phi_0^* \rangle_X = 1$ . By Lemma 1.3,

$$\frac{d\lambda}{d\mu} \Big|_{\mu=\mu_c} = \left\langle \frac{d\mathcal{A}_\mu}{d\mu} \Big|_{\mu=\mu_c} \phi_0, \phi_0^* \right\rangle_X.$$

Taking the real part gives

$$\alpha'(\mu_c) = \operatorname{Re} \left\langle \frac{d\mathcal{A}_\mu}{d\mu} \Big|_{\mu=\mu_c} \phi_0, \phi_0^* \right\rangle_X.$$

By assumption,

$$\alpha'(\mu_c) \neq 0.$$

Therefore, the critical eigenvalues cross the imaginary axis transversely as  $\mu$  passes through  $\mu_c$ . The spectral decomposition associated with the critical pair yields

$$X = X_c \oplus X_s,$$

More precisely,

$$X_c = \text{span} \{\phi_0, \overline{\phi_0}\}.$$

When the system is expressed in real coordinates, the center subspace has real dimension two,  $X_c \simeq \mathbb{R}^2$ . Because the operator is not necessarily self-adjoint, the associated spectral projection is a Riesz spectral projection, not in general an orthogonal projection; denote it by.

$$P_c: X \rightarrow X_c, \quad P_s = I - P_c.$$

The spectral gap

$$\text{Re } \lambda \leq -\delta < 0$$

$$\mathcal{M}_c = \{x + h(x, \mu): x \in B_\varepsilon(0) \subset X_c, |\mu - \mu_c| < \varepsilon\}, \quad h: X_c \times \mathbb{R} \rightarrow X_s,$$

$$\text{and } h(0, \mu_c) = 0, \quad D_x h(0, \mu_c) = 0.$$

The dynamics on the center manifold can be represented by a two-dimensional system.

$$\begin{aligned} \frac{dx}{dt} &= \begin{pmatrix} \alpha(\mu) & -\omega(\mu) \\ \omega(\mu) & \alpha(\mu) \end{pmatrix} x + H(x, \mu), \\ H(x, \mu) &= O(\|x\|^2). \end{aligned}$$

At the critical parameter,  $\alpha(\mu_c) = 0$ ,  $\omega(\mu_c) = \omega_0$ . Since  $\alpha'(\mu_c) \neq 0$ . Denoting the corresponding solution on the center manifold by  $x_\mu(t)$ , the solution in the full Hilbert space is reconstructed through

$$U_\mu(t) = U^* + x_\mu(t) + h(x_\mu(t), \mu).$$

The period satisfies

$$T(\mu) = \frac{2\pi}{\omega_0} [1 + O(|\mu - \mu_c|)] \quad \text{as } \mu \rightarrow \mu_c.$$

Thus, the infinite-dimensional reaction-diffusion system shows a local Hopf bifurcation at  $\mu_c$ . The resulting periodic orbit may be interpreted as an infinite-dimensional analogue of a limit cycle.

### Example 1

We now illustrate the main theoretical results with an explicit example on the one-dimensional domain

$$\begin{aligned} \Omega &= (0, \pi) \subset \mathbb{R}, \\ D(x) &= D_0 = 0.1. \end{aligned}$$

Consider the linearized operator  $\mathcal{L}_\mu$  and

$$\begin{aligned} X &= L^2(0, \pi) \times L^2(0, \pi) \\ D(\mathcal{L}_\mu) &= \{(u, v) \in H^2(0, \pi) \times H^2(0, \pi): u'(0) = u'(\pi) = 0, v'(0) = v'(\pi) = 0\}. \end{aligned}$$

The Neumann eigenfunctions of the scalar diffusion operator  $-D_0 \frac{d^2}{dx^2}$  are

$$\phi_0(x) = \frac{1}{\sqrt{\pi}}, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \cos(kx), \quad k \geq 1,$$

and

$$\lambda_k^{(D)} = D_0 k^2 = 0.1k^2, \quad k \in \mathbb{N}_0.$$

We choose the parameter values

$$a = 0.1, \quad \varepsilon = 0.01, \quad \mu_c = 0.5.$$

Consider the cubic nonlinearity

$$f(u) = u(1-u)(u-0.1) = -u^3 + 1.1u^2 - 0.1u.$$

Its derivative is

$$f'(u) = -3u^2 + 2.2u - 0.1.$$

The critical equilibrium  $u^*$  is determined by

$$f'(u^*) = \varepsilon \mu_c = 0.005.$$

Thus,

$$\begin{aligned} -3(u^*)^2 + 2.2u^* - 0.1 &= 0.005, \\ -3(u^*)^2 + 2.2u^* - 0.105 &= 0. \end{aligned}$$

and

$$u^* = \frac{2.2 \pm \sqrt{3.58}}{6}.$$

The smaller equilibrium is

$$u^* = \frac{2.2 - \sqrt{3.58}}{6} \approx 0.051318.$$

At this equilibrium,  $f'(u^*) = 0.005$ .

For each spatial mode  $k \in \mathbb{N}_0$ , the corresponding  $2 \times 2$  block of the linearized operator is

$$J_k(\mu) = \begin{pmatrix} -D_0 k^2 + f'(u^*) & -1 \\ \varepsilon & -\varepsilon \mu \end{pmatrix}.$$

At the critical parameter  $\mu_c = 0.5$ ,

$$J_k(\mu_c) = \begin{pmatrix} -0.1k^2 + 0.005 & -1 \\ 0.01 & -0.005 \end{pmatrix}.$$

The eigenvalues of each mode are determined by

$$\lambda^2 - \text{tr}(J_k) \lambda + \det(J_k) = 0.$$

For  $k = 0$ ,

$$J_0 = \begin{pmatrix} 0.005 & -1 \\ 0.01 & -0.005 \end{pmatrix}.$$

Its trace is  $\text{tr}(J_0) = 0$ ,

$$\det(J_0) = (0.005)(-0.005) - (-1)(0.01) = 0.009975.$$

The characteristic discriminant is

$$\Delta_0 = \text{tr}(J_0)^2 - 4 \det(J_0) = -0.0399 < 0.$$

So, the eigenvalues are purely imaginary:

$$\begin{aligned} \lambda_0 &= \pm i\sqrt{0.009975}, \\ \lambda_0 &\approx \pm 0.099875 i. \end{aligned}$$

Thus,

$$\omega_0 = \sqrt{0.009975} \approx 0.099875.$$

The corresponding linear oscillation period is approximately

$$T_0 = \frac{2\pi}{\omega_0} \approx 62.92.$$

For  $k = 1$ ,

$$J_1 = \begin{pmatrix} -0.095 & -1 \\ 0.01 & -0.005 \end{pmatrix}.$$

Thus  $\text{tr}(J_1) = -0.1$ ,  $\det(J_1) = 0.010475$ .

The discriminant is

$$\Delta_1 = (-0.1)^2 - 4(0.010475) = -0.0319 < 0.$$

Then,

$$\begin{aligned} \lambda_1 &= \frac{-0.1 \pm i\sqrt{0.0319}}{2}, \\ \lambda_1 &\approx -0.05 \pm 0.089303 i. \end{aligned}$$

Since,  $\text{Re } \lambda_1 = -0.05 < 0$ , For  $k \geq 2$ , the diffusion term

$$-D_0 k^2 = -0.1k^2$$

For  $k = 2$ ,  $\text{tr}(J_2) = -0.4$  and

$$\Delta_2 = (-0.4)^2 - 4(0.011975) = 0.1121 > 0,$$

the eigenvalues are real and distinct:

$$\lambda_{2,1} \approx -0.03256, \quad \lambda_{2,2} \approx -0.36744.$$

Both eigenvalues are strictly negative.

For  $k = 3$ ,  $\text{tr}(J_3) = -0.9$  and

$$\Delta_3 = 0.81 - 4(0.014475) = 0.7521 > 0.$$

So,  $\lambda_{3,1} \approx -0.01638$ ,  $\lambda_{3,2} \approx -0.88362$ .

Then, both eigenvalues lie strictly in the left half-plane. For larger  $k$ , the term  $-0.1k^2$  becomes increasingly dominant, ensuring increasingly strong diffusive damping of the corresponding high-frequency spatial modes. The preceding calculations show that, at  $\mu = \mu_c = 0.5$ ,

$$\sigma(\mathcal{L}_{\mu_c}) \cap i\mathbb{R} = \{\pm 0.099875 i\}.$$

The first nonhomogeneous mode has the stable complex eigenvalues

$$-0.05 \pm 0.089303 i,$$

Consequently,

$$\sigma(\mathcal{L}_{\mu_c}) \setminus \{\pm i\omega_0\} \subset \{z \in \mathbb{C}: \text{Re } z < 0\}.$$

Thus, the only spectral modes that reach the imaginary axis at the critical parameter are the spatially homogeneous modes. The higher spatial modes stay linearly stable because of the increasing diffusive contribution  $-D_0 k^2$ . To verify the crossing condition explicitly, consider the homogeneous mode  $k = 0$ . For general  $\mu$ ,

$$J_0(\mu) = \begin{pmatrix} f'(u^*) & -1 \\ \varepsilon & -\varepsilon\mu \end{pmatrix}.$$

Such that  $f'(u^*)$  is independent of  $\mu$  in the local linearization, its trace is

$$\text{tr } J_0(\mu) = f'(u^*) - \varepsilon\mu,$$

at the critical value,

$$\begin{aligned} f'(u^*) &= \varepsilon\mu_c, \\ \text{tr } J_0(\mu_c) &= 0. \end{aligned}$$

For the complex conjugate pair,

$$\begin{aligned} \lambda(\mu) &= \alpha(\mu) \pm i\omega(\mu), \\ \alpha(\mu) &= \frac{1}{2} \text{tr } J_0(\mu). \end{aligned}$$

It follows that

$$\alpha'(\mu_c) = -\frac{\varepsilon}{2}.$$

Since  $\varepsilon = 0.01$ , so,

$$\alpha'(\mu_c) = -0.005 \neq 0.$$

Thus, the spectral transversality condition is satisfied. Because  $\alpha'(\mu_c) = -0.005 \neq 0$  and  $\lambda(\mu_c) = \pm i\omega_0$ , where  $\omega_0 \approx 0.099875$ , the critical eigenvalue pair crosses the imaginary axis transversely. This causes the homogeneous equilibrium to change from unstable to stable as  $\mu$  increases past  $\mu_c = 0.5$ . According to the infinite-dimensional Hopf theorem, a local branch of nontrivial periodic solutions emerges at  $\mu_c$ , with periods satisfying

$$T(\mu) \rightarrow \frac{2\pi}{0.099875} \approx 62.92 \quad \text{as } \mu \rightarrow \mu_c.$$

In the full Hilbert space, these solutions take the form

$$U_\mu(t) = U^* + x_\mu(t) + h(x_\mu(t), \mu),$$

representing sustained oscillations whose stability is not guaranteed by transversality alone and requires future analysis.

## Conclusion and Future Directions

The coupled reaction-diffusion system admits unique mild solutions provided suitable dissipativity and regularity conditions hold, and the spectral properties of its linearized non-self-adjoint operator are characterized. As the parameter  $\mu$  varies, a simple conjugate pair of eigenvalues crosses the imaginary axis for the spatial mode ( $k = 0$ ), while the higher modes ( $k \geq 1$ ) remain strictly in the left half-plane. Together with the required spectral and regularity assumptions, the transversality  $\alpha'(\mu_c) \neq 0$  guarantees an infinite-dimensional Hopf bifurcation and nontrivial periodic solutions. These oscillatory dynamics describe sustained periodic behavior relevant to ionic and membrane potential instabilities in myocardial ischemia [1, 5, 17]. Future work may consider higher-dimensional domains, spatially varying coefficients and interactions involving multiple critical eigenvalues that may lead to higher-order bifurcations.

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