

Traveling Wave Solutions of the Burgers, Huxley and Nagumo Equations via the Variation of (G'/G) -Expansion Method: Analytical Fronts for Nerve Impulse Propagation in Excitable Media

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Abstract

The Burgers–Huxley and Nagumo equations are canonical bistable reaction-diffusion models for the propagation of an action potential along a nerve fiber, distinguished from the classical Fisher–KPP equation by a cubic reaction term with three equilibria: a resting state, an excitation threshold, and an excited state. This paper applies the Variation of the (G'/G) -Expansion Method to both equations. For the Burgers–Huxley equation, under both the standard wave transformation $\xi = x - ct$ and a general transformation $\xi = rx + vt$, the method returns only the two trivial constant equilibria $u = 1$ and $u = k$, indicating that first-order balancing is insufficient to recover a genuine connecting front for this closure. For the Nagumo equation with an additional nonlinear convection term, the same method applied with the general transformation $\xi = rx + vt$ yields twenty parameter families and thirty-eight explicit closed-form solutions. Critical examination of these solutions shows that exactly three of the tanh-type families are bounded, monotone kink solutions connecting each pair of the equilibria $\{0, a, 1\}$ of the underlying cubic nonlinearity, reproducing the classical bistable-front structure of the Nagumo equation, while the coth-type and composite families contain a non-removable singularity and are therefore mathematically valid but not physiologically realizable as impulses. These results clarify which of the many solutions produced by (G'/G) -type expansion methods correspond to genuine traveling nerve-impulse fronts, and illustrate the role of the convection and threshold parameters in setting the front speed and steepness.

Keywords. Traveling Wave Solutions, (G'/G) -Expansion Method, Burgers huxley Equation, Nagumo Equation, Bistable Reaction diffusion, Nerve Impulse Propagation, Excitable Media

Introduction

The propagation of an action potential along a nerve fiber is one of the paradigmatic examples of a traveling front in an excitable medium. Following the reduction of the Hodgkin–Huxley model by FitzHugh (1961) and its electronic realization by Nagumo, Arimoto and Yoshizawa (1962), a large family of scalar reaction–diffusion equations with a cubic, bistable reaction term has been used to isolate the essential front-propagation behavior of nerve impulses from the full ionic-channel kinetics. Two closely related members of this family are considered here: the Burgers–Huxley equation, which combines Burgers-type nonlinear advection with a Huxley-type cubic reaction term and was first solved analytically by Wang, Zhu and Lu (1990), and the Nagumo equation with nonlinear diffusion and convection, a further generalization in which the advective term is explicitly retained alongside the classical bistable cubic reaction $u(1 - u)(u - a)$.

Exact traveling-wave solutions of these equations are valuable for at least three reasons: they provide closed-form benchmarks against which numerical schemes can be validated, they make the dependence of front speed and steepness on the physiological parameters (threshold a , convection strength α) analytically transparent, and because the underlying cubic nonlinearity has three equilibria (a resting state $u = 0$, a threshold $u = a$, and an excited state $u = 1$) they can reveal which heteroclinic connections between these states are realized by a given closure. The (G'/G) -expansion method of Wang, Li and Zhang (2008) and its many subsequent extensions have been applied extensively to the Burgers–Huxley family; Zhu (2016), for example, used the basic (G'/G) -expansion method to construct solitary and kink solutions of the Burgers–Huxley equation directly.

The present study uses a two-function variant of this family, the Variation of the (G'/G) -expansion method, in which the wave amplitude is expanded jointly in the Riccati-type quantities G'/G and F'/F . We apply it first to the Burgers–Huxley equation under both the simplest wave transformation and a general transformation, and then to the richer Nagumo equation with nonlinear diffusion and convection, which, owing to its larger parameter set, admits twenty distinct solution families. A central contribution of this paper is not merely to enumerate these families, as is common in the (G'/G) -expansion literature, but to classify them by physical admissibility: which are bounded, monotone fronts connecting genuine equilibria of the reaction term, and which contain mathematical singularities that preclude a direct physiological interpretation.

Section 2 summarizes the method. Section 3 applies it to the Burgers–Huxley equation. Section 4 applies it to the Nagumo equation with nonlinear diffusion and convection and classifies the resulting solution families. Section 5 discusses the physiological interpretation of the results, and Section 6 concludes.

The Variation of the (G'/G)-Expansion Method

For a nonlinear PDE $P(u, u_t, u_x, u_{xx}, \dots) = 0$, the method proceeds through six steps, summarized here for completeness (a full derivation is given in the companion paper on Murray's equation using the same technique).

Step 1. Apply the traveling-wave transformation $u(x, t) = U(\xi)$, with $\xi = rx + vt$ (or the special case $\xi = x - ct$), reducing the PDE to an ODE in $U(\xi)$.

Step 2. Integrate the ODE term-by-term where possible.

Step 3. Expand the solution as

$$U(\xi) = \sum_{i=0}^m a_i (G'/G)^i + \sum_{i=1}^m b_i (G'/G)^{i-1} (F'/F) \quad (1)$$

where G and F satisfy the coupled Riccati-type system $G' = -GF$, $F' = 1 - F^2$, whose known closed-form solutions are the hyperbolic pairs

$(G, F) = (\pm \operatorname{sech} \xi, \tanh \xi)$ and $(\pm \operatorname{csch} \xi, \coth \xi)$.

Step 4. Fix the balancing order m using the homogeneous-balance rule applied to the highest-order derivative and the highest-order nonlinear term.

Step 5. Substitute the ansatz into the ODE, collect the coefficients of each power of $F(\xi)$, and set each coefficient to zero, producing an over determined nonlinear algebraic system.

Step 6. Solve the algebraic system with a computer algebra system (Maple) and substitute the resulting constants, together with the closed-form Riccati solutions, back into the ansatz to obtain the explicit traveling-wave solution(s).

Traveling Wave Solutions of the Burgers–Huxley Equation

Standard Wave Transformation ($\xi = x - ct$)

The Burgers–Huxley equation considered here is

$$u_t - u_{xx} - uu_x - u(k - u)(u - 1) = 0 \quad (2)$$

where the cubic reaction term $u(k - u)(u - 1)$ has equilibria at $u = 0, k, 1$, generalizing the standard Nagumo cubic $u(1 - u)(u - a)$ by allowing an independent upper threshold k . With $u(x, t) = U(\xi)$, $\xi = x - ct$, Eq. (2) reduces to

$$cU' + U'' + uU' + U(k - U)(U - 1) = 0 \quad (3)$$

Balancing U'' against U^3 gives $m + 2 = 3m$, i.e. $m = 1$, so the ansatz reduces to

$$U(\xi) = a_0 - a_1 F + a_2 F^2 + b_1 (F^{-1} - F) \quad (4)$$

Substituting (4) into (3) and collecting the coefficients of F^i ($i = 0, \pm 1, \dots, 6$) produces a ten-equation algebraic system (Appendix A, Eqs. A.1–A.10). Solving with Maple gives two non-trivial constant solution families,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad c = c \quad (5)$$

$$a_0 = k, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad c = c \quad (6)$$

Corresponding to the two constant traveling-wave solutions

$$u_1(x, t) = 1, \quad u_2(x, t) = k \quad (7)$$

General Wave Transformation ($\xi = rx + vt$)

To test whether the restriction to unit-speed transformation $\xi = x - ct$ is responsible for the absence of a connecting front, the method is reapplied under the general transformation $u(x, t) = U(\xi)$, $\xi = rx + vt$, with independent constants r and v . Equation (2) reduces to

$$r^2 U'' + (rU - v)U' - U(k - U)(U - 1) = 0 \quad (8)$$

The same balancing ($m = 1$) and ansatz (4) apply. Substitution and collection of the coefficients of F^i ($i = 0, \pm 1, \dots, 6$) again produces a ten-equation system (Appendix A, Eqs. A.11–A.20), whose non-trivial solutions are, once more,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad r = r, \quad v = v \quad (9)$$

$$a_0 = k, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad r = r, \quad v = v \quad (10)$$

$$u_1(x, t) = 1, \quad u_2(x, t) = k \quad (11)$$

The identity of results (7) and (11) under two structurally different wave transformations confirms that the outcome is a genuine property of the first-order ansatz (4), not an artifact of the specific transformation chosen.

Interpretation

Both wave transformations return only the two constant equilibria $u = 1$ and $u = k$ of the cubic reaction term, with no monotone front connecting them. This is consistent with the balancing order obtained here, $m = 1$: ansatz (4) has only one degree of hyperbolic-function freedom (a single power of F), which is sufficient to represent a constant solution but not a genuine kink connecting two distinct equilibria; such a kink generically requires the amplitude to pass through an odd combination of F and F^{-1} terms that cancels only for special coefficient choices. As Section 4 shows, retaining the full quadratic term structure (as in the extended Nagumo equation) restores the ability of the same method to recover bounded connecting fronts. The Burgers–Huxley results obtained here should therefore be read as a first-order benchmark useful for confirming the two physical equilibria and for validating the algebraic-reduction procedure rather than as the complete solution set of the equation, which is known from the literature (e.g., Wang, Zhu & Lu, 1990) to include monotone kink solutions obtainable via higher-order or non-integer-power auxiliary-equation methods.

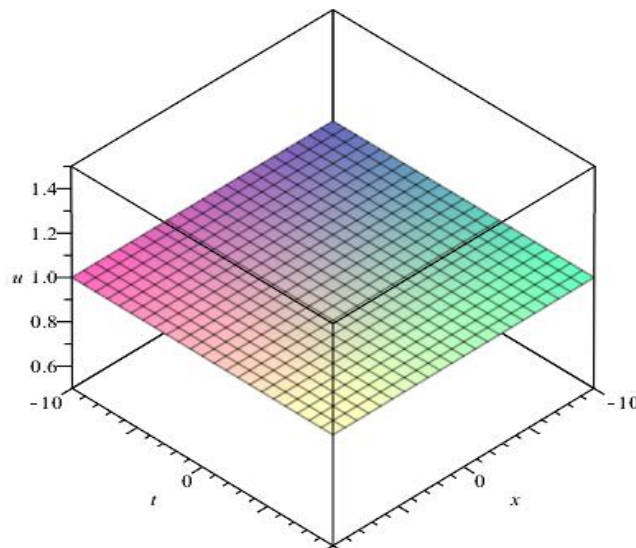


Figure 1. Surface plot of the constant equilibrium solution $u_1(x, t) = 1$ of the Burgers–Huxley equation (Eq. 7).

Traveling Wave Solutions of the Nagumo Equation with Nonlinear Diffusion and Convection

Governing Equation, Balancing and Ansatz

The Nagumo equation with nonlinear diffusion and convection is

$$u_t - u_{xx} + \alpha u u_x - u(1-u)(u-a) = 0 \quad (12)$$

where α controls the strength of the nonlinear convective term and $a \in (0, 1)$ is the excitation threshold; the cubic reaction term $u(1-u)(u-a)$ has the same three equilibria structure as the Burgers–Huxley closure, at $u = 0$ (rest), $u = a$ (threshold) and $u = 1$ (excited state). With the general transformation $u(x, t) = U(\xi)$, $\xi = rx + vt$, Eq. (12) reduces to

$$r^2 U'' + (-v - \alpha r U) U' + U(1-U)(U-a) = 0 \quad (13)$$

Balancing U'' against U^3 gives $m = 1$, so

$$U(\xi) = a_0 - a_1 F + a_2 F^2 + b_1(F^{-1} - F) \quad (14)$$

Substituting (14) into (13) and collecting the coefficients of F^i ($i = 0, \pm 1, \dots, 6$) produces a ten-equation algebraic system (Appendix A, Eqs. A.21–A.30). Unlike the Burgers–Huxley case, the presence of the independent convection parameter α and the two free transformation constants r, v allows Maple to return twenty distinct non-trivial parameter families (Table 1), each of which generates one or two explicit closed-form solutions via substitution into (14) together with the hyperbolic pair (7)–(8) of the auxiliary equations, for a total of thirty-eight explicit traveling-wave solutions.

Table 1. Parameter families (Cases 1–20) obtained by solving the algebraic system for the Nagumo equation with nonlinear diffusion and convection ($a_2 = 0$ in every case). Each row generates a Type-1 (tanh) and Type-2 (coth) solution via ansatz (14).

Case	a_0	a_1	b_1	R	v
1	1	0	0	R	v
2	A	0	0	R	v
3	$a/2$	$a/2$	0	$(a/8)(\alpha\sqrt{(\alpha^2+8)})$	$a/2 - a^2/4 - (\alpha a^2/16)(\alpha\sqrt{(\alpha^2+8)})$
4	$a/2$	$-a/2$	0	$(a/8)(-\alpha\sqrt{(\alpha^2+8)})$	$a^2/4 - a/2 - (\alpha a^2/16)(-\alpha\sqrt{(\alpha^2+8)})$
5	$1/2$	$1/2$	0	$(\alpha\sqrt{(\alpha^2+8)})/8$	$a/2 - 1/4 - (\alpha/16)(\alpha\sqrt{(\alpha^2+8)})$
6	$1/2$	$-1/2$	0	$(-\alpha\sqrt{(\alpha^2+8)})/8$	$1/4 - a/2 - (\alpha/16)(-\alpha\sqrt{(\alpha^2+8)})$
7	$(a+1)/2$	$(a-1)/2$	0	$((a-1)/8)(\alpha\sqrt{(\alpha^2+8)})$	$(\alpha/16)(\alpha\sqrt{(\alpha^2+8)}) + 1/4 - (\alpha a^2/16)(\alpha\sqrt{(\alpha^2+8)}) - a^2/4$
8	$(a+1)/2$	$(1-a)/2$	0	$((a-1)/8)(-\alpha\sqrt{(\alpha^2+8)})$	$(\alpha/16)(-\alpha\sqrt{(\alpha^2+8)}) - 1/4 - (\alpha a^2/16)(-\alpha\sqrt{(\alpha^2+8)}) + a^2/4$
9	$a/2$	$-a/2$	$a/2$	$(a/8)(-\alpha\sqrt{(\alpha^2+8)})$	$a^2/4 - a/2 - (\alpha a^2/16)(-\alpha\sqrt{(\alpha^2+8)})$
10	$a/2$	$-a/2$	$a/4$	$(a/16)(-\alpha\sqrt{(\alpha^2+8)})$	$a^2/8 - a/4 - (\alpha a^2/32)(-\alpha\sqrt{(\alpha^2+8)})$
11	$a/2$	$a/2$	$-a/2$	$(a/8)(\alpha\sqrt{(\alpha^2+8)})$	$a/2 - a^2/4 - (\alpha a^2/16)(\alpha\sqrt{(\alpha^2+8)})$
12	$a/2$	$-a/2$	$-a/4$	$(a/16)(\alpha\sqrt{(\alpha^2+8)})$	$a/2 - a^2/4 - (\alpha a^2/32)(\alpha\sqrt{(\alpha^2+8)})$
13	$a/2$	$-a/2$	$1/2$	$(a/8)(-\alpha\sqrt{(\alpha^2+8)})$	$1/4 - a/2 - (\alpha/16)(-\alpha\sqrt{(\alpha^2+8)})$
14	$1/2$	$1/2$	$-1/2$	$(\alpha\sqrt{(\alpha^2+8)})/8$	$a/2 - 1/4 - (\alpha/16)(\alpha\sqrt{(\alpha^2+8)})$
15	$1/2$	$-1/2$	$1/4$	$(-\alpha\sqrt{(\alpha^2+8)})/16$	$1/8 - a/4 - (\alpha/32)(-\alpha\sqrt{(\alpha^2+8)})$
16	$1/2$	$1/2$	$-1/4$	$(\alpha\sqrt{(\alpha^2+8)})/16$	$a/4 - 1/8 - (\alpha/32)(\alpha\sqrt{(\alpha^2+8)})$
17	$(a+1)/2$	$(1-a)/2$	$(a-1)/2$	$((a-1)/8)(-\alpha\sqrt{(\alpha^2+8)})$	$a^2/4 + (\alpha/16)(-\alpha\sqrt{(\alpha^2+8)}) - (\alpha a^2/16)(-\alpha\sqrt{(\alpha^2+8)}) - 1/4$

18	$(a+1)/2$	$(1-a)/2$	$(a-1)/4$	$((a-1)/16)(-\alpha\sqrt{(\alpha^2+8)})$	$(\alpha/32)(-\alpha\sqrt{(\alpha^2+8)}) - 1/8 - (\alpha a^2/32)(-\alpha\sqrt{(\alpha^2+8)}) + a^2/8$
19	$(a+1)/2$	$(a-1)/2$	$(1-a)/2$	$((a-1)/8)(\alpha\sqrt{(\alpha^2+8)})$	$(\alpha/16)(\alpha\sqrt{(\alpha^2+8)}) + 1/4 - (\alpha a^2/16)(\alpha\sqrt{(\alpha^2+8)}) - a^2/4$
20	$(a+1)/2$	$(a-1)/2$	$(1-a)/4$	$((a-1)/16)(\alpha\sqrt{(\alpha^2+8)})$	$(\alpha/32)(\alpha\sqrt{(\alpha^2+8)}) + 1/8 - (\alpha a^2/32)(\alpha\sqrt{(\alpha^2+8)}) - a^2/8$

Representative Closed-Form Solutions

Substituting the Case 1–8 parameter values of (Table 1) into ansatz (14) together with the hyperbolic auxiliary solutions gives the following explicit traveling-wave solutions (Type 1 uses $F = \tanh \xi$; Type 2 uses $F = \coth \xi$, with $\xi = rx + vt$ and the (r, v) of the corresponding case).

Constant equilibria (Cases 1–2).

$$u_1(x, t) = 1, \quad u_2(x, t) = a \quad (15)$$

Front connecting $u = 0$ and $u = a$ (Cases 3–4).

$$u_3(x, t) = a/2 - (a/2) \tanh[(a/8)(\alpha \pm \sqrt{(\alpha^2+8)})x + (a/2 - a^2/4 - (\alpha a^2/16)(\alpha \pm \sqrt{(\alpha^2+8)}))t] \quad (16)$$

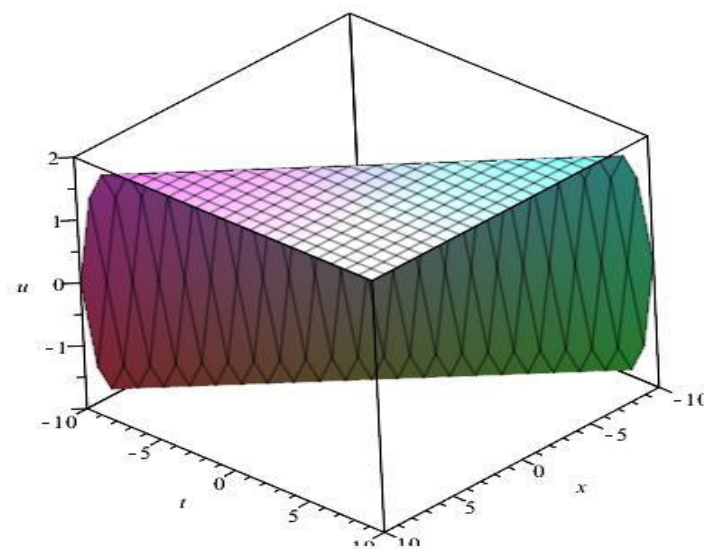
Front connecting $u = 0$ and $u = 1$ (Cases 5–6).

$$u_7(x, t) = 1/2 - (1/2) \tanh[(\alpha \pm \sqrt{(\alpha^2+8)})/8 x + (a/2 - 1/4 - (\alpha/16)(\alpha \pm \sqrt{(\alpha^2+8)}))t] \quad (17)$$

Front connecting $u = a$ and $u = 1$ (Cases 7–8).

$$u_{11}(x, t) = (a+1)/2 - ((a-1)/2) \tanh[(a-1)/8(\alpha \pm \sqrt{(\alpha^2+8)})x + (\dots)t] \quad (18)$$

Each of Eqs. (16)–(18) is a smooth, monotone, bounded kink that interpolates between two of the three equilibria of the cubic reaction term as ξ ranges over $\mathbb{R}$, with the sign choice ($\pm$) and the sign of the argument controlling the direction of propagation and steepness of the front. Cases 9–20 (Table 1) generate composite solutions of the form $A \pm B \tanh(\cdot) \pm C \coth(\cdot)$, obtained by the same substitution; their explicit forms follow directly from (Table 1) and are omitted here for brevity but are recoverable by direct substitution into (14).



(Figure 2). Representative bounded kink-front solution of the Nagumo equation with nonlinear diffusion and convection (Type-1, tanh family; Case 5/6, Eq. 17), shown as a function of x and t .

Classification by Physical Admissibility

Not all thirty-eight formal solutions generated by (Table 1) are equally admissible as models of a propagating nerve impulse. The function $\tanh \xi$ is smooth and bounded, $\tanh \xi \in (-1, 1)$, for all real ξ , so every Type-1 (tanh-only) solution Eqs. (15)–(18) and their Case 9–20 analogues restricted to the tanh part are globally bounded, monotone traveling fronts. In contrast, $\coth \xi$ has a non-removable pole at $\xi = 0$ and diverges as $\xi \rightarrow 0$, so every Type-2 (coth-only) solution and every composite solution containing a coth term is singular along the moving line $rx + vt = 0$ and is therefore unbounded on $\mathbb{R}$. Such solutions remain mathematically valid solutions of the reduced ODE away from the singularity, and are of interest for completeness of the solution family and for possible interpretation as local (rather than global) approximations, but they cannot represent a physiologically realizable action-potential profile, which must remain bounded for all x and t .

Applying this criterion to (Table 1), exactly three physically admissible bounded kink families are obtained: Cases 3/4, 5/6, and 7/8, and they connect, respectively, the three possible pairs among the equilibria $\{0, a, 1\}$: rest-to-threshold, rest-to-excited, and threshold-to-excited. This is precisely the heteroclinic structure expected of a bistable cubic reaction term, and its recovery here is a non-trivial check on the correctness of the algebraic reduction, since the (G'/G) -expansion procedure does not build in this structure a priori; it emerges from the solved algebraic system. The remaining Type-2 and

composite families (Cases 9–20 and the Type-2 branches of Cases 3–8) enlarge the formal solution set but do not correspond to additional physiological wave fronts.

Results and Discussion

Two related but structurally different outcomes emerge from this study. For the Burgers–Huxley equation, the first-order ansatz recovers only the two constant equilibria of the cubic reaction term, under both the standard and the general wave transformation; no connecting front is produced at this balancing order. For the Nagumo equation with nonlinear diffusion and convection, the richer parameter structure (an independent convection coefficient α and two free transformation constants r, v) allows the same method to generate twenty parameter families, of which three tanh-type families reproduce the complete set of bounded heteroclinic fronts connecting the equilibria of the cubic nonlinearity the classical bistable-front picture of excitable-medium theory.

The wave speed and steepness of each bounded front are set jointly by the threshold parameter a and the convection coefficient α through the expressions for r and v in (Table 1): increasing $|\alpha|$ increases the magnitude of the argument of tanh (through the $\alpha \pm \sqrt{\alpha^2 + 8}$ factor), steepening the front, while the threshold a enters both the amplitude prefactor and the argument, shifting the balance between diffusive spreading and advective sharpening. This is qualitatively consistent with the known result that the front speed of the classical (convection-free) Nagumo equation is set solely by the threshold parameter, $c = \sqrt{2} (1/2 - a)$, with convection introducing an additional, tunable contribution to both speed and steepness. The explicit identification of which solutions are singular (Section 4.3) is, in our view, a necessary complement to the routine application of (G'/G)-type expansion methods in the biological-modeling literature, where large families of formal solutions are often reported without an explicit boundedness check. We suggest that this classification step, testing each closed-form family for poles of the auxiliary functions over the physical domain, should be standard practice whenever such methods are applied to equations, like the Nagumo and Burgers–Huxley equations, whose physical solutions are required to be globally bounded.

Limitations of the present analysis include the restriction to a first-order balancing throughout (the Burgers–Huxley cubic term forces $m = 1$, precluding richer ansatz forms without a modified balancing rule), and the absence of a formal stability analysis of the three bounded fronts identified in Section 4.3; both are natural directions for follow-up work, alongside a numerical comparison of front speeds against direct simulation of Eq. (12).

Conclusion

The Variation of the (G'/G)-expansion method was applied to the Burgers–Huxley equation and to the Nagumo equation with nonlinear diffusion and convection, two canonical bistable models of nerve impulse propagation. For the Burgers–Huxley equation, the method at the balancing order dictated by its cubic nonlinearity returns only the two constant equilibria of the reaction term under two different wave transformations. For the Nagumo equation, the larger parameter space yields twenty families and thirty-eight explicit solutions, of which exactly three tanh-type families are globally bounded and reproduce the complete set of heteroclinic fronts connecting the resting, threshold and excited states of the underlying bistable kinetics; the remaining coth-type and composite families are shown to be singular and therefore not physiologically realizable. Beyond the specific results obtained, this study illustrates the importance of a systematic physical-admissibility check for the large solution families that (G'/G)-type expansion methods routinely produce, particularly in biological applications where boundedness is a physical requirement rather than a mathematical nicety.

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Appendix A. Full Algebraic Systems

All systems below were obtained by substituting the respective ansatz into the reduced ODE, collecting the coefficients of each power of $F(\xi)$, and solving with Maple.

A.1 Burgers–Huxley Equation, $\xi = x - ct$ (Eqs. A.1–A.10)

$$F^6: -a_2^3 = 0 \quad (\text{A.1})$$

$$F^5: 3a_1a_2^2 + 3a_2^2b_1 - 2a_2^2 = 0 \quad (\text{A.2})$$

$$F^4: -3a_0a_2^2 - 3a_1^2a_2 - 6a_1a_2b_1 + a_2^2k - 3a_2b_1^2 + 3a_1a_2 + a_{22} + 3a_2b_1 + 6a_2 = 0 \quad (\text{A.3})$$

$$F^3: 6a_0a_1a_2 + 6a_0a_1b_1 + a_1^3 + 3a_1^2b_1 + 2a_2^2 - 2aa_2b_1 - 2a_2c - b_1^2 - 2a_1 - 2b_1 = 0 \quad (\text{A.4})$$

$$F^2: -3a_0^2a_2 - 3a_0a_1^2 - 6a_0a_1b_1 + 2a_0a_2k - 3a_0b_1^2 + a_1^2k + 6a_1a_2b_1 + 2a_1b_1k + 6a_2b_1^2 + b_1^2k + a_0a_1 + 2a_0a_2 + a_0b_1 + a_1^2 - 3a_1a_2 + 2a_1b_1 + a_1c - 4a_2b_1 - a_2k + b_1^2 + b_1c - 8a_2 = 0 \quad (\text{A.5})$$

$$F^1: 3a_0^2a_1 + 3a_0^2b_1 - 2a_0a_1k - 6a_0a_2b_1 - 2a_0b_1k - 3a_1^2b_1 - 6a_1b_1^2 + 2a_2b_1k - 3b_1^3 - 2a_0a_1 + 2a_0a_2 - 2a_0b_1 + a_1^2 + 2a_1b_1 + a_1k + 2a_2b_1 + 2a_2c + b_1^2 + b_1k + 2a_1 + 2b_1 = 0 \quad (\text{A.6})$$

$$F^0: -a_0^3 + a_0^2k + 6a_0a_1b_1 + 6a_0b_1^2 - 2a_1b_1k - 3a_2b_1^2 - 2b_1^2k + a_0^2 - a_0a_1 - a_0k - 2a_1b_1 - a_1c + a_2b_1 - 2b_1^2 + 2a_2 = 0 \quad (\text{A.7})$$

$$F^{-1}: -3a_0b_1^2 + 2a_0b_1k + 3a_1b_1^2 + 3b_1^3 + 2a_0b_1 + b_1^2 - b_1k - 2b_1 = 0 \quad (\text{A.8})$$

$$F^{-2}: -3a_0b_1^2 + b_1^2k - a_0b_1 + b_1^2 - b_1c = 0 \quad (\text{A.9})$$

$$F^{-3}: -b_1^3 - b_1^2 + 2b_1 = 0 \quad (\text{A.10})$$

A.2 Burgers–Huxley Equation, $\xi = rx + vt$ (Eqs. A.11–A.20)

$$F^6: -a_2^3 = 0 \quad (\text{A.11})$$

$$F^5: 3a_1a_2^2 + 3a_2^2b_1 - 2a_2^2 + r = 0 \quad (\text{A.12})$$

$$F^4: -3a_0a_2^2 - 3a_1^2a_2 - 6a_1a_2b_1 + 3a_1a_2r + a_2^2k - 3a_2b_1^2 + 3b_1r + 6a_2r^2 + a_2^3 = 0 \quad (\text{A.13})$$

$$F^3: 6a_0a_1a_2 + 6a_0a_2b_1 - 2a_0a_2r + a_1^3 + 3a_1^2b_1 - a_1^2r - 2a_1a_2k + 3a_1b_1^2 - 2a_1b_1r - 2a_1r^2 - 3a_2^2b_1 + 2a_2^2r - 2a_2b_1k + b_1^3 - b_1^2r - 2b_1r^2 - 2a_1a_2 - 2a_2b_1 + 2a_2v = 0 \quad (\text{A.14})$$

$$F^2: -3a_0^2a_2 - 3a_0a_1^2 - 6a_0a_1b_1 + a_0a_1r + 2a_0a_2k - 3a_0b_1^2 + a_0b_1r + a_1^2k + 6a_1a_2b_1 - 3a_1a_2r + 2a_0a_1b_1k + 6a_2b_1^2 - 4a_2b_1r - 8a_2r^2 + b_1^2k + 2a_0a_2 + a_1^2 + 2a_1b_1 - a_1v - a_2k + b_1^2 - b_1v = 0 \quad (\text{A.15})$$

$$F^1: 3a_0^2a_1 + 3a_0^2b_1 - 2a_0a_1k - 6a_0a_2b_1 + 2a_0a_2r - 2a_0b_1k - 3a_1^2b_1 + a_1^2r - 6a_1b_1^2 + 2a_1b_1r + 2a_1r^2 + 2a_2b_1k - 3b_1^3 + b_1^2r + 2b_1r^2 - 2a_0a_1 - 2a_0b_1 + a_1k + 2a_2b_1 - 2a_2v + b_1k = 0 \quad (\text{A.16})$$

$$F^0: -a_0^3 + a_0^2k + 6a_0a_1b_1 - a_0a_1r + 6a_0b_1^2 - 2a_1b_1k - 3a_2b_1^2 + a_2b_1r + 2a_2b_1r^2 - 2b_1^2k + a_0^2 - a_0k - 2a_1b_1 + a_1v - 2b_1^2 = 0 \quad (\text{A.17})$$

$$F^{-1}: -3a_0b_1^2 + 2a_0b_1k + 3a_1b_1^2 + 3b_1^3 + b_1^2r - 2b_1r^2 + 2a_0b_1 - b_1k = 0 \quad (\text{A.18})$$

$$F^{-2}: -3a_0b_1^2 - a_0b_1r + b_1^2k + b_1^2 + b_1v = 0 \quad (\text{A.19})$$

$$F^{-3}: -b_1^3 - b_1^2r + 2b_1r^2 = 0 \quad (\text{A.20})$$

A.3 Nagumo Equation with Nonlinear Diffusion and Convection (Eqs. A.21–A.30)

$$F^6: a_2^2 = 0 \quad (\text{A.21})$$

$$F^5: 2a_2^2\alpha r + 3a_1a_2^2 + 3a_2^2b_1 = 0 \quad (\text{A.22})$$

$$F^4: -3a_1a_2\alpha r - 3b_1a_2r + a_2^2 - 3a_0a_2^2 - 3a_1^2a_2 - 6a_1a_2b_1 - 3a_2b_1^2 + 6a_2r^2 + a_2^2 = 0 \quad (\text{A.23})$$

$$F^3: 2a_0a_2\alpha r + a_2^2\alpha r + 2a_1a_2\alpha r - 2a_2^2\alpha r + ab_1^2r - 2\alpha a_1a_2 - 2\alpha a_2b_1 + 6a_0a_1a_2 + 6a_0a_2b_1 + a_1^3 + 3a_1^2b_1 - 2a_1r^2 - 3a_2^2b_1 + b_1^3 - 2b_1r^2 - 2a_1a_2 - 2b_1a_2 + 2a_2v = 0 \quad (\text{A.24})$$

$$F^2: a_0a_1\alpha r - a_0\alpha b_1r + 3a_1a_2\alpha r + 4a_2\alpha b_1r + 2\alpha a_0a_2 + \alpha a_1^2 + 2\alpha a_1b_1 + \alpha b_1^2 - 3a_0^2a_2 - 3a_0a_1^2 - 6a_0a_1b_1 - 3a_0b_1^2 + 6a_1a_2b_1 + 6a_2b_1^2 - 8a_2r^2 - \alpha a_2 + 2a_0a_2 + a_1^2 + 2a_1b_1 - a_1v + b_1^2 - b_1v = 0 \quad (\text{A.25})$$

$$F^1: -2a_0a_2\alpha r - a_1^2\alpha r - 2a_1\alpha b_1r - \alpha a_2b_1^2r - 2\alpha a_0a_1 - 2\alpha a_0b_1 + 2\alpha a_2b_1 + 3a_0^2a_1 + 3a_0^2b_1 - 6\alpha a_0a_2b_1 - 3a_1^2b_1 - 6a_1b_1^2 + 2a_1r^2 - 3b_1^3 + 2b_1r^2 + \alpha a_1 + \alpha b_1 - 2a_0a_1 - 2a_0b_1 + 2a_2b_1 - 2a_2v = 0 \quad (\text{A.26})$$

$$F^0: a_0a_1\alpha r - a_2\alpha b_1r + \alpha a_0^2 - 2\alpha a_1b_1 - 2\alpha b_1^2 - a_0^3 + 6a_0a_1b_1 + 6a_0b_1^2 - 3a_2b_1^2 + 2a_2r^2 - \alpha a_0 + a_0^2 - 2a_1b_1 + a_1v - 2b_1^2 = 0 \quad (\text{A.27})$$

$$F^{-1}: -\alpha b_1^2r + 2\alpha a_0b_1 - 3a_0^2b_1 + 3a_1b_1^2 + 3b_1^3 - 2b_1r^2 - \alpha b_1 + 2a_0b_1 = 0 \quad (\text{A.28})$$

$$F^{-2}: a_0\alpha b_1r + \alpha b_1^2 - 3a_0b_1^2 + b_1^2 + b_1v = 0 \quad (\text{A.29})$$

$$F^{-3}: \alpha b_1^2r - b_1^3 + 2b_1r^2 = 0 \quad (\text{A.30})$$