

Original article

Measurement-Noise-Adaptive Sliding-Mode Observer for Sparse State Estimation of a Nonlinear Distillation Column

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Abstract

Reliable state estimates are needed when important process variables are not measured directly. This paper develops a measurement-noise-adaptive sliding-mode observer (MN-ASMO) to estimate unmeasured distillate and bottoms compositions in a 32-state binary distillation column. The observer estimates the measurement noise level during operation and uses this estimate to adjust both the boundary-layer thickness and the dead-zone threshold that controls gain adaptation. MN-ASMO is compared to a fixed-gain SMO (FG-SMO), a residual-adaptive SMO (RA-SMO), and an extended Kalman filter with adaptive measurement covariance (R-adaptive EKF), using four internal composition measurements and 30 paired Monte Carlo trials across four scenarios. Between 5 and 80 minutes, MN-ASMO reduces the mean composition root-mean-square error (RMSE) relative to RA-SMO by 15.5%, 23.7%, and 6.4% under increased noise, model mismatch, and combined uncertainty, respectively, but increases the nominal RMSE by 10.0%. The individual and combined effects of the two noise-dependent updates are assessed while all other observer settings remain unchanged. Under increased measurement noise, using both updates reduces composition RMSE by 53.5% compared with keeping both settings fixed and yields lower RMSE than either update alone. Under combined uncertainty, the joint update reduces error variation but does not clearly outperform the individual updates in RMSE. The increased noise scenario produces a noise level similar to that of FG-SMO. These results show that joint adaptation should be considered when measurement noise varies, although the benefits depend on the operating conditions.

Keywords. Distillation Column; Sliding-Mode Observer; Residual Adaptation; Measurement-Noise Adaptation.

Introduction

Accurate composition estimates are important for monitoring and controlling distillation. If measurements are only available at a few internal stages, a dynamic model can help estimate the unmeasured distillate and bottoms compositions. Estimation is more difficult when the plant differs from the nominal model or when measurement noise changes [1–5]. Researchers have explored model-based composition estimation using various observer designs. Lang and Gilles [1] developed nonlinear observers for distillation columns, while Baratti et al. [2] evaluated a binary-column composition estimator in a pilot plant. Later studies investigated extended Kalman filtering for batch distillation [3], geometrically validated observers [4], and state estimation in high-dimensional chemical processes [5]. Alexander et al. [6] reviewed broader challenges in nonlinear estimation for chemical and biochemical processes. Recent research integrates process knowledge with measured data. Examples include hybrid models applying neural-network soft sensors [7], soft sensors trained on refinery operating data [8], and hybrid estimators for distillation-in-series plants [9]. Additional studies utilize limited local measurements to reconstruct column behavior [10], estimate product compositions in heat-integrated columns [11], or combine adaptive regression with constrained Kalman estimators [12]. These investigations use diverse models and measurement configurations.

The present study considers a mechanistic binary-column model with four direct measurements of internal liquid composition. Changing measurement noise can affect estimators that adjust their corrections online. Extended Kalman filtering and moving-horizon estimation are established approaches to nonlinear state estimation [13]. Adaptive Kalman methods update noise covariance estimates during operation [14–16], while robust EKF formulations address Gaussian and non-Gaussian measurement errors in chemical processes [17].

Sliding-mode observers (SMOs) use output-error injection to estimate nonlinear process states [18,19]. Ríos et al. [20] developed an adaptive sliding-mode observer for simultaneous state and parameter estimation. Franco et al. [21] examined joint state and time-varying parameter estimation in the presence of measurement noise and disturbances. Variable-bandwidth extended state observers also address noise sensitivity by adjusting observer bandwidth to balance estimation performance and noise tolerance [22]. These studies motivate examining how measurement noise should influence residual-driven gain adaptation.

In a residual-driven regularized SMO, the residual contains both estimation error and measurement noise, so increased noise can activate gain growth. The boundary-layer thickness controls the local correction slope at a given gain, while the dead-zone threshold determines when the filtered residual activates the gain-growth term. The specific question addressed here is whether using an online noise estimate to adjust both settings improves unmeasured-composition estimation beyond either update alone. We investigate this question for the binary-column benchmark using a nominal process model and bounded adaptive correction gains.

This work develops a measurement-noise-adaptive sliding-mode observer (MN-ASMO) that uses an online noise estimate to adjust both the boundary-layer thickness and the dead-zone threshold in the residual-driven gain law. These updates aim to limit the influence of measurement noise on state correction and gain growth when estimating unmeasured distillate and bottoms compositions. We describe the sampled implementation, including initialization, update timing, numerical

integration, and parameter bounds, and provide an algorithm summary. We compare MN-ASMO with FG-SMO, RA-SMO, and an R-adaptive EKF using 30 paired Monte Carlo trials for each scenario. The evaluation covers nominal operation, changing measurement noise, and model parameter uncertainty. A matched component study examines four configurations with neither update enabled, only the boundary-layer update, only the dead-zone update, or both updates. All other observer settings and measurement records remain identical across these configurations. We report composition RMSE, paired confidence intervals, error variation, and computation time to identify when the joint update is beneficial and assess any trade-offs in nominal accuracy, error variation, or computational cost.

Methods

Nonlinear Distillation-Column Model

We use the binary distillation benchmark described by Hahn and Edgar [23], based on the column studied by Horton et al. [24]. The model includes 30 equilibrium trays, a total condenser, and a reboiler, for 32 liquid-composition states. State 1 is the condenser, state 32 is the reboiler, and the feed enters at state 17. Figure 1 shows the state numbering and measurement locations.

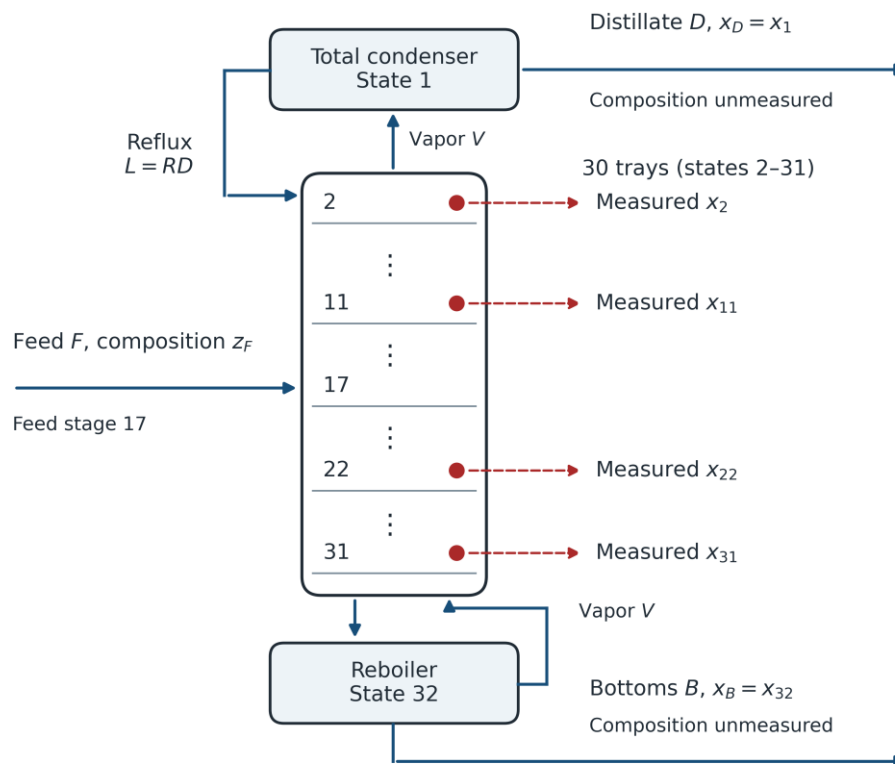


Figure 1. Schematic of the 32-state binary distillation column used in the benchmark. The four measured liquid compositions are at stages 2, 11, 22, and 31; the feed enters at stage 17, and the distillate and bottoms compositions are not measured directly.

The state vector contains 32 liquid-composition states, ordered from the condenser to the reboiler:

$$\mathbf{x}(t) = [x_1(t) \ x_2(t) \ \cdots \ x_{32}(t)]^T \in \mathbb{R}^{32}. \quad (1)$$

The input vector is

$$\mathbf{u}(t) = [R(t) \ F(t) \ z_F(t)]^T, \quad (2)$$

where R is the reflux ratio, F is the feed flow rate, and z_F is the feed composition. The nominal relative volatility is $\alpha = 1.6$. Here, x_i denotes the liquid-phase mole fraction of the light component at stage i of the binary distillation column. The vapor-liquid equilibrium relation is

$$y_i = \frac{\alpha x_i}{1 + (\alpha - 1)x_i}. \quad (3)$$

Its analytical derivative with respect to the liquid-phase mole fraction is

$$\frac{\partial y_i}{\partial x_i} = \frac{\alpha}{[1 + (\alpha - 1)x_i]^2}. \quad (4)$$

The flow relations are

$$D = \frac{F}{2}, \quad L = RD, \quad (5)$$

$$V = L + D, \quad (6)$$

$$L_s = L + F, \quad (7)$$

$$B = F - D. \quad (8)$$

D and B denote the distillate and bottoms flow rates, respectively. L and L_s are the liquid flow rates above and below the feed stage, respectively, and V is the vapor flow rate. All flow rates are in mol/min.

The dynamic component balances are

$$M_1 \frac{dx_1}{dt} = V(y_2 - x_1), \quad (9)$$

$$M_i \frac{dx_i}{dt} = L(x_{i-1} - x_i) + V(y_{i+1} - y_i), \quad i = 2, \dots, 16, \quad (10)$$

$$M_{17} \frac{dx_{17}}{dt} = Lx_{16} + Fz_F - L_s x_{17} + V(y_{18} - y_{17}), \quad (11)$$

$$M_i \frac{dx_i}{dt} = L_s(x_{i-1} - x_i) + V(y_{i+1} - y_i), \quad i = 18, \dots, 31, \quad (12)$$

$$M_{32} \frac{dx_{32}}{dt} = L_s x_{31} - Bx_{32} - Vy_{32}. \quad (13)$$

The benchmark assumes constant liquid holdups, relative volatility, and molar overflow within each column section [23, 24], a saturated-liquid feed, and no energy, pressure, or analyzer dynamics. The nominal condenser, tray, and reboiler holdups are 0.5, 0.25, and 0.1 mol, respectively. M_i denotes the liquid holdup at stage i . The plant and observer are both integrated with fourth-order Runge-Kutta. Ten plant substeps and five observer substeps are used within the 0.05 min sampling interval.

Measurements and sampling time

Four liquid-composition states, x_2 , x_{11} , x_{22} , and x_{31} , are measured, while the complete 32-state composition vector is estimated. The condenser and reboiler compositions, x_1 and x_{32} , are not measured and are used as unmeasured evaluation variables. The product compositions at the condenser and reboiler are deliberately left unmeasured because they are the quantities of interest for evaluating the observers.

$$\mathbf{y}_k = C\mathbf{x}_k + \mathbf{v}_k, \quad (14)$$

where $C \in \mathbb{R}^{4 \times 32}$ selects those four states, and $\mathbf{v}_k$ is measurement noise.

The condenser and reboiler product compositions are not measured. They are estimated and used only for evaluation. The information flow is strictly causal: the estimate reported at t_k has used measurements only through t_{k-1} ; the current measurement $\mathbf{y}_k$ is used to compute the correction and propagate the observer to t_{k+1} . The observer does not have access to the true noise schedule, hidden feed-composition variation, or perturbed plant parameters.

Common Sliding-Mode Observer Structure

The three SMO implementations share the output-injection structure motivated by [18, 19]. We define the injection distribution for this benchmark below. At the nominal steady state, we first verify that all eigenvalues of the local plant Jacobian A have negative real parts. We then solve the continuous-time Lyapunov equation using the standard linear-system construction [25].

$$AW + WA^T = -I. \quad (15)$$

The measurement-distribution matrix is constructed as

$$D = \text{diag} \left(\frac{1}{\text{diag}(CWC^T)} \right), \quad (16)$$

$$G = WC^T D. \quad (17)$$

For all SMO variants, the correction is driven by the innovation residual

$$\mathbf{r}_k = \mathbf{y}_k - C\hat{\mathbf{x}}_k. \quad (18)$$

Using the standard scalar saturation term [18], the continuous boundary-layer observer is written as

$$\dot{\hat{\mathbf{x}}}(t) = f(\hat{\mathbf{x}}(t), u_k, p) + \sum_{j=1}^4 g_j \left[k_p e_{j,k}(t) + K_{j,k} \text{sat} \left(\frac{e_{j,k}(t)}{\varphi_{j,k}} \right) \right]. \quad (19)$$

Here, g_j is the j th column of G , and $e_{j,k}(t)$ is the j th component of the residual $e_k(t) = z_k - C\hat{\mathbf{x}}(t)$. The scalar parameters $K_{j,k}$ and $\varphi_{j,k} > 0$ are the correction gain and boundary-layer thickness for measurement channel j , respectively. The common proportional gain k_p is scalar, and $\text{sat}(\cdot)$ is the scalar saturation function defined in Eq. (20). Each bracketed term is scalar, so multiplication by g_j gives a 32-dimensional state correction.

$$\text{sat}(q) = \begin{cases} -1, & q < -1, \\ q, & -1 \leq q \leq 1, \\ 1, & q > 1. \end{cases} \quad (20)$$

The input u_k , measurement z_k , updated gain K_k , and layer thickness ϕ_k remain constant during the sampling interval. We recompute $e_k(t)$ at each RK4 stage. The common proportional gain is $k_p = 0.5 \text{ min}^{-1}$. Saturation makes the correction continuous within a finite boundary layer; we therefore describe these implementations as regularized SMOs. The Lyapunov equation defines a local design matrix and does not establish global convergence of the nonlinear sampled observer.

Fixed gain sliding mode observer

The fixed-gain sliding-mode observer (FG-SMO) uses the standard SMO structure [18, 19], with all channel gains set to 0.012 and the boundary-layer thickness set to 0.003. This benchmark implementation serves as a reference with no online gain or noise adaptation.

Residual-Adaptive Sliding-Mode Observer (R-ASMO)

The residual-adaptive sliding-mode observer (RA-SMO) retains the output-injection structure [18, 19] and updates its four gains using the filtered measurement residual. Its boundary layer and dead-zone threshold remain fixed. The following gain law defines the adaptive reference used in this study. The parameter-adaptive observers in [20, 21] provide related background, but their gain laws and convergence results do not define or establish convergence for this sampled implementation.

For the measurement channel j , the residual is filtered according to

$$\mu_{j,k} = (1 - a_r)\mu_{j,k-1} + a_r r_{j,k}, \quad (21)$$

where

$$a_r = 1 - \exp\left(-\frac{T_s}{T_r}\right). \quad (22)$$

The switching gain is updated by

$$K_{j,k+1} = \Pi_{[K_{\min}, K_{\max}]}(K_{j,k} + T_s[\gamma q_{j,k} - \lambda(K_{j,k} - K_{\min})]). \quad (23)$$

Here $\Pi_{[K_{\min}, K_{\max}]}$ denotes component-wise projection of the switching gain onto the admissible interval.

$$q_{j,k} = \max(|\mu_{j,k}| - \delta_j, 0). \quad (24)$$

RA-SMO uses $\phi = 0.003$ and a fixed threshold $\delta_0 \approx 4.4703 \times 10^{-4}$. This threshold equals $2\sigma_0$ times the residual filter's white-noise standard-deviation ratio, derived below. The adaptation rate is $\gamma = 8$. The threshold multiplier of 2.0 is fixed during both rate selection and evaluation and is independent of the MN-ASMO multiplier. When the filtered residual is within the dead zone, leakage remains active and drives the gain toward its lower bound.

Causal Measurement-Noise Estimation

For each measured channel, we calculate the second difference $\Delta^2 z_{j,k} = z_{j,k} - 2z_{j,k-1} + z_{j,k-2}$. This removes a locally linear trend in the noise-free composition signal. If the measurement noise is independent and Gaussian with locally constant variance σ^2 , the noise contribution to the second difference has variance $(1^2 + 2^2 + 1^2)\sigma^2 = 6\sigma^2$. We apply the median absolute deviation (MAD), a robust measure of scale [26], to these second differences:

$$\hat{\sigma}_{j,k}^{\text{raw}} = \frac{\text{MAD}(\Delta^2 y_j)}{0.67448975\sqrt{6}} \quad (25)$$

Here $\text{MAD}(v) = \text{median}(|v - \text{median}(v)|)$. Both medians use the second differences available in the trailing measurement window. The Gaussian MAD conversion factor is 0.67448975 [26]; the additional $\sqrt{6}$ in Eq. (25) accounts for the variance of the second-difference noise. We bound the raw estimate and apply exponential smoothing:

$$\hat{\sigma}_{j,k} = (1 - a_\sigma)\hat{\sigma}_{j,k-1} + a_\sigma \min\{\sigma_{\max}, \max\{\sigma_{\min}, \hat{\sigma}_{j,k}^{\text{raw}}\}\}, \quad (26)$$

where

$$a_\sigma = 1 - \exp\left(-\frac{T_s}{T_\sigma}\right). \quad (27)$$

The buffer stores at most 80 measurements, corresponding to a 4-min window. Noise estimation begins after 25 measurements, which provide 23-second differences. Until then, the estimate remains at $\sigma \approx \sigma_0 = 0.001$. We use $T_\sigma = 0.25$ min, $\sigma_{\min} = 0.0005$, and $\sigma_{\max} = 0.03$. All calculations use only measurements available at the current sample. Signal curvature, correlated noise, and a variance change within the window can affect the estimate, so it serves as a noise-level indicator under the stated assumptions.

Proposed measurement-Noise-Adaptive Sliding-Mode Observer (MN-ASMO)

The proposed MN-ASMO combines the regularized SMO structure [18,19] with the MAD-based noise estimate [26]. Its distinguishing feature is using this estimate in both the boundary-layer and dead-zone updates. The residual filter and bounded gain law remain as defined above. Equations (28)–(31) specify the coupling proposed here; the evaluated gain-adaptation rate is $\gamma_N = 30$.

The current noise estimate sets the boundary-layer thickness:

$$\phi_{j,k} = \min\{\phi_{\max}, \max\{\phi_0, m_\phi \hat{\sigma}_{j,k}\}\}. \quad (28)$$

It also sets the dead-zone threshold in the gain-adaptation law:

$$\delta_{j,k} = \min\left\{\delta_{\max}, m_\delta \hat{\sigma}_{j,k} \sqrt{\frac{a_r}{2-a_r}}\right\}. \quad (29)$$

The switching gain follows

$$K_{j,k+1} = \Pi_{[K_{\min}, K_{\max}]}(K_{j,k} + T_s[\gamma_N q_{j,k} - \lambda(K_{j,k} - K_{\min})]), \quad (30)$$

where now

$$q_{j,k} = \max(|\mu_{j,k}| - \delta_{j,k}, 0). \quad (31)$$

The distinction from R-ASMO can be summarised as follows:

$$\text{R-ASMO: } r_k \rightarrow K_k, \quad (32)$$

$$\text{MN-ASMO: } r_k \rightarrow K_k, \quad \hat{\sigma}_k \rightarrow \phi_k, \delta_k. \quad (33)$$

The MN-ASMO multipliers are $m_\phi = m_\delta = 1.5$, and both upper limits, $\phi_{\max}$ and $\delta_{\max}$, are 0.02. For the first-order residual filter driven by independent white noise, the steady-state output-to-input standard-deviation ratio is $\sqrt{a_r/(2 - a_r)}$. This follows from the variance recursion $s^2 = (1 - a_r)^2 s^2 + a_r^2 \sigma^2$. We use this ratio to scale the dead zone, while recognizing that feedback and model mismatch also affect the residual. At a fixed gain, a wider layer reduces the local saturation slope. A larger dead zone reduces gain growth from small filtered residuals; leakage remains active. The component study evaluates these effects separately and jointly.

Algorithm 1. Online Procedure for MN-ASMO

Initialize the state estimate as specified in the simulation setup. Set all four filtered residuals to zero, each observer gain to 0.025, and each noise-scale estimate to σ_0 . Store the most recent measurements from each channel, keeping no more than 80 samples per channel.

1. At sample k , store the prior state estimate for performance evaluation. Acquire the measurement z_k and the known input u_k .
2. Append z_k to the measurement buffer. Once 25 measurements are available, calculate, bound, and smooth the noise estimate using Eqs. (25)–(27). Until then, retain σ_0 .
3. Calculate the sampled and filtered residuals using Eqs. (18), (21), and (22). Update the layer thickness and dead zone using Eqs. (28) and (29).
4. Update and bound each channel gain using Eqs. (30) and (31). The evaluated configuration uses $\gamma_N = 30$.
5. Hold z_k , u_k , the updated gains, and the layer thicknesses fixed. Integrate Eq. (19) using five RK4 substeps of length $T_s/5$, and recompute $e_k(t)$ at each RK4 stage.
6. If needed, project each propagated state to $[0, 1]$. Use the result as the prior estimate at sample $k + 1$ and repeat from step 1.

EKF with adaptive measurement covariance (R-Adaptive EKF)

The R-adaptive EKF uses standard EKF prediction and measurement updates [13,25]. It computes the measurement covariance R_k using the same causal noise estimate as MN-ASMO. The adaptive filtering studies [14–16] and robust process-estimation formulation provide related background; the covariance rule below is the benchmark implementation used here:

$$R_k = \eta \text{diag}(\hat{\sigma}_{1,k}^2, \hat{\sigma}_{2,k}^2, \hat{\sigma}_{3,k}^2, \hat{\sigma}_{4,k}^2). \quad (34)$$

The innovation covariance and gain are

$$S_k = C P_k C^T + R_k, \quad (35)$$

$$L_k = P_k C^T S_k^{-1}, \quad (36)$$

and the measurement update is written as

$$\hat{\mathbf{x}}_k^{\text{post}} = \hat{\mathbf{x}}_k + L_k \mathbf{r}_k. \quad (37)$$

The covariance uses the Joseph form:

$$P_k^{\text{post}} = (I - L_k C) P_k (I - L_k C)^T + L_k R_k L_k^T. \quad (38)$$

The covariance multiplier is $\eta = 1$. We predict the state with the RK4 map and evaluate its analytical Jacobian F_k at the updated state estimate:

$$\hat{\mathbf{x}}_{k+1}^- = F_{RK4}(\hat{\mathbf{x}}_k^+, u_k) \quad (39)$$

$$P_{k+1}^- = F_k P_k^+ F_k^T + q T_s I \quad (40)$$

Here, F_k is the Jacobian of the RK4 transition map. The process-noise density is $q = 3 \times 10^{-5}$, and the initial covariance is $0.03^2 I$. State estimates are projected to $[0, 1]$ as needed. The projection was inactive in every reported trial, so the equations above describe the updates evaluated.

Simulation Setup

We evaluate the observers using prescribed reflux-ratio and feed-flow trajectories without a feedback controller. Four internal liquid compositions are measured, while the distillate and bottoms compositions are not measured. Tables 1 and 2 list the process and observer settings. Each trial lasts 80 min.

Uncertainty scenarios

Each trial uses an amplitude factor a_u drawn uniformly from $[0.9, 1.1]$ and an event-time offset b_u drawn uniformly from $[-0.6, 0.6]$ min. The reflux ratio is $3 - 0.5a_u$ on $[10 + b_u, 25 + b_u]$, $3 + 0.5a_u$ on $[25 + b_u, 45 + b_u]$, and $3 - 0.2a_u$ on $[45 + b_u, 60 + b_u]$, with time in minutes. It is 3 otherwise. Feed flow is $2 + 0.2a_u$ mol/min on $[30 + b_u, 50 + b_u]$ min and 2 mol/min otherwise. Within

each trial, all observers use the same randomized input signals and the same noisy measurement records to ensure a paired comparison. The four scenarios are:

Nominal: nominal plant parameters and constant noise standard deviation $\sigma_0 = 0.001$.

Increased noise: nominal plant parameters; noise standard deviation is 0.01 during $20 \leq t < 40$ min and 0.006 during $55 \leq t < 70$ min, and σ_0 otherwise.

Model mismatch: relative volatility is drawn uniformly within $\pm 5\%$ of its nominal value, and each stage's holdup is drawn independently within $\pm 10\%$. The true feed composition changes from 0.5 to $0.5 + 0.04a_u$ during $35 + b_u \leq t < 65 + b_u$ min. All observers continue to use $z_F = 0.5$, and the noise standard deviation remains σ_0 .

Combined uncertainty: the same mismatch structure together with the increased-noise schedule.

Each plant starts at its own steady state. All methods within a trial share the same initial estimate. For stage i , this estimate is $0.85x_{i,ss} + 0.075 + b_0 \sin(i\pi/33)$, where $x_{i,ss}$ is the nominal steady-state composition and $i = 1, \dots, 32$. The perturbation amplitude b_0 is drawn uniformly from $[-0.01, 0.01]$.

Table 1. Main process and simulation settings.

Parameter	Value	Description
Number of states	32	30 trays + condenser + reboiler
Feed stage	17	Saturated-liquid feed
Relative volatility, α	1.6	Nominal value
Holdups	0.5; 0.25; 0.1 mol	Condenser; trays 2–31; reboiler
Nominal input $[R, F, z_F]^T$	$[3, 2, 0.5]^T$	Reflux ratio, feed flow, feed composition
Sensor stages	2, 11, 22, 31	Four internal liquid-composition measurements
Sampling interval, T_s	0.05 min (3 s)	Common observer sampling interval
Simulation duration	80 min	All scenarios
Nominal sensor noise, σ_0	0.001	Mole-fraction standard deviation
Increased-noise windows	20–40 min; 55–70 min	$10\sigma_0$; $6\sigma_0$, respectively
Evaluation interval	5–80 min	Initial observer transient excluded
Training seeds	101–103	Used only for tuning
Evaluation seeds	4001–4030	30 paired trials/scenario

Observer Configurations and Tuning

Preliminary training fixed the FG-SMO gain, noise-estimator settings, noise-update multipliers, and EKF process-noise density. We then refined the RA-SMO and MN-ASMO adaptation rates using the same candidates: 5, 8, 12, 20, 30, 45, and 60. We evaluated each candidate on 12 training records, comprising seeds 101–103 across the four scenarios. The objective was the mean composition RMSE from 5 to 80 min, with equal weight assigned to each record. We selected $\gamma = 8$ and $\gamma_N = 30$, both within the tested range. We froze all settings before evaluation and used seeds 4001–4030, separate from the training seeds.

The standard SMO settings are $T_r = 0.5$ min, $k_p = 0.5 \text{ min}^{-1}$, gain bounds $[0.003, 0.15]$, leakage $\lambda = 0.4 \text{ min}^{-1}$, and $\phi_0 = 0.003$. Adaptive gains start at 0.025, and filtered residuals start at zero. The final rate searches use the same candidate set and objective, though preliminary tuning differs across methods. The comparisons present the selected configurations and do not establish globally optimal tuning.

Table 2. Observer configurations and final parameters selected from independent training data.

Method	Adaptation mechanism	Final parameters
FG-SMO	Fixed gain; fixed boundary layer	$K = 0.012$; $\phi = 0.003$
RA-SMO	Residual-driven gain adaptation; fixed layer and dead zone	$\gamma = 8$; dead-zone multiplier = 2.0; $\phi = 0.003$
MN-ASMO	Residual-driven gains; noise-dependent layer and dead zone	$\gamma_N = 30$; $m\phi = m\delta = 1.5$; $T\sigma = 0.25$ min; $\phi_{\max} = \delta_{\max} = 0.02$
R-adaptive EKF	Adaptive measurement covariance R_k	$q = 3 \times 10^{-5}$; $\eta = 1$; $P_0 = 0.03^2 I$

The RA-SMO dead-zone multiplier remains 2.0 throughout training and evaluation; MN-ASMO uses its own multiplier of 1.5. Consequently, their main comparison includes differences in adaptation rate and nominal dead-zone setting. The controlled component study below holds these settings constant to assess the noise-dependent updates themselves.

Controlled component study

Four variants isolate the two noise-dependent updates. Neither update keeps both settings fixed; Layer adapts only the boundary-layer thickness; Threshold adapts only the dead-zone threshold; Both applies the full MN-ASMO update. All four retain residual-driven gain adaptation with $\gamma_N = 30$ and share initialization, gain limits, leakage, multipliers, and integration settings. A disabled update is fixed at the value of its own law evaluated at σ_0 . These values are $\phi = 0.003$ and $\delta = 3.3527 \times 10^{-4}$.

The variant with neither noise-dependent update is therefore a residual-adaptive observer, not FG-SMO. The variants are not retuned individually; this study tests component effects within the selected MN-ASMO configuration. They use the same 30 input and measurement records per scenario as the main comparison. The Both trajectory is the primary MN-ASMO trajectory and is reused without another simulation. The five tested contrasts compare Both with each of the other three variants and compare each individual update with Neither update.

Performance measures and statistical analysis

The product compositions are the mole fractions of the more-volatile component in the distillate and bottoms streams. Their estimation errors are $e_{D,k} = \hat{x}_{1,k} - x_{1,k}$ and $e_{B,k} = \hat{x}_{32,k} - x_{32,k}$. Let K contain samples with $5 \leq t_k \leq 80$ min and let $N_p = |K|$. The primary composition RMSE, denoted ProductRMSE in the implementation, aggregates these two unmeasured states:

$$\text{ProductRMSE} = \sqrt{\frac{1}{2N} \sum_{k \in K} [(x_D - \hat{x}_D)^2 + (x_B - \hat{x}_B)^2]}. \quad (41)$$

These stress windows apply in every scenario and fall within the increased-noise periods when that schedule is active. Table 3 additionally reports RMSE across all 32 states over 5–80 min. This all-state measure is distinct from the two-product metric in Eq. (41). To describe how rapidly the product-estimation errors vary, we calculate

$$\text{ErrorRateRMS} = \sqrt{\frac{1}{2N_d} \sum_k \left[\left(\frac{\Delta e_D}{T_s} \right)^2 + \left(\frac{\Delta e_B}{T_s} \right)^2 \right]}, \quad (42)$$

Here $\Delta e_k = e_{k+1} - e_k$. The set K' contains intervals whose two endpoints are at or after 5 min, and $N_d = |K'|$. Product RMSE is expressed as a mole fraction; ErrorRateRMS is expressed in mole fraction/min. ErrorRateRMS describes temporal variation of the error. It is not a formal chattering index and should be interpreted alongside estimation accuracy.

For the RA-SMO comparison, the paired composition RMSE difference in evaluation trial i is

$$\Delta_k = \text{ProductRMSE}_{R\text{-ASMO},k} - \text{ProductRMSE}_{MN\text{-ASMO},k}. \quad (43)$$

A positive d_i indicates a lower error for MN-ASMO. The mean paired difference is

$$\bar{\Delta} = \frac{1}{n} \sum_{k=1}^n \Delta_k, \quad (44)$$

The two-sided 95% confidence interval for the mean paired difference is:

$$\bar{\Delta} \pm t_{0.975, n-1} \frac{s_{\Delta}}{\sqrt{n}}. \quad (45)$$

Here, $n = 30$, and sd is the sample standard deviation of the paired differences. The same calculation applies to the other reference methods, component variants, and reported metrics, with all differences computed as reference minus candidate. We report two-sided paired t -tests with Holm-adjusted p -values [27]. We apply the adjustment separately within each metric and scenario, using three contrasts for the main comparison and five for the component study.

Results and Discussion

Numerical Verification and Evaluation Completion

The main comparison includes 480 observer trajectories: four methods, four scenarios, and 30 trials. The component study adds three variants per trial and reuses the full MN-ASMO trajectory, yielding 840 distinct evaluated trajectories. No observer failures, state projections, or computation times exceeding the 3 s sampling interval were recorded. The analytical plant Jacobian and RK4 transition Jacobian have relative errors of 3.09×10^{-11} and 2.64×10^{-10} , respectively, and the mass-balance residual is 2.78×10^{-16} . Component checks also confirm the causal noise calculation, the independence of the RA-SMO threshold, and agreement between the full method and its component-study counterpart.

Composition estimation

Under nominal conditions, RA-SMO gives the lowest mean composition RMSE, 1.531×10^{-4} , compared with 1.684×10^{-4} for MN-ASMO (Table 3). MN-ASMO has higher error in all 30 trials, corresponding to a 10.0% increase in the mean (Table 4).

Under increased noise, MN-ASMO reduces mean composition RMSE by 15.5% relative to RA-SMO and shows lower error in all 30 trials. Its stress-window RMSE is 21.5% lower than that of RA-SMO. However, the R-adaptive EKF yields the lowest stress-window value, 7.34×10^{-4} , whereas MN-ASMO yields 1.03×10^{-3} . Thus, the ranking over the full 5–80 min record differs from the ranking within the increased-noise periods.

MN-ASMO and FG-SMO have nearly identical mean accuracy in the increased-noise scenario. The mean paired difference, calculated as FG-SMO minus MN-ASMO, is 5.42×10^{-7} , with a 95% confidence interval of $[-1.74 \times 10^{-5}, 1.85 \times 10^{-5}]$ and a Holm-adjusted p -value of 0.951. These trials do not establish an accuracy advantage for MN-ASMO over FG-SMO; do not interpret the interval containing zero as proof of equivalence.

Under model mismatch, the mean composition RMSE decreases from 2.210×10^{-3} for RA-SMO to 1.686×10^{-3} for MN-ASMO, a 23.7% reduction. Under combined uncertainty, it decreases from 2.946×10^{-3} to 2.757×10^{-3} , a 6.4% reduction. MN-ASMO has lower error in all 30 trials across both scenarios. These findings apply to the specified variations in relative volatility, holdup, and feed composition.

Figure 2 illustrates the first held-out trial under combined uncertainty, using seed 4001. Figure 3 shows the trial-to-trial spread across all scenarios. The larger spread under model mismatch and combined uncertainty reflects sensitivity to the randomized plant parameters. Table 4 reports paired differences comparing methods on the same plant and measurement record, rather than treating their errors as independent samples.

Table 3. Mean errors over 30 paired trials per scenario. RMSE values and sample standard deviations are shown in units of 10^{-3} mole fraction. Composition RMSE over 5–80 min is the primary metric; all-state RMSE covers all 32 states.

Scenario	Method	Composition RMSE	Std. dev.	Stress-window RMSE	All-state RMSE
Nominal	R-adaptive EKF	0.318	0.008	0.312	0.306
Nominal	FG-SMO	0.239	0.008	0.249	0.359
Nominal	RA-SMO	0.153	0.008	0.150	0.235
Nominal	MN-ASMO	0.168	0.009	0.164	0.256
Noise	R-adaptive EKF	0.773	0.047	0.734	0.842
Noise	FG-SMO	0.741	0.040	1.056	1.192
Noise	RA-SMO	0.877	0.054	1.316	1.419
Noise	MN-ASMO	0.741	0.074	1.033	1.151
Mismatch	R-adaptive EKF	2.359	1.622	2.359	7.008
Mismatch	FG-SMO	4.034	3.197	4.034	9.724
Mismatch	RA-SMO	2.210	1.066	2.220	6.939
Mismatch	MN-ASMO	1.686	0.911	1.679	6.571
Combined	R-adaptive EKF	4.605	3.057	6.119	9.970
Combined	FG-SMO	4.571	3.117	5.064	9.998
Combined	RA-SMO	2.946	1.104	3.585	7.415
Combined	MN-ASMO	2.757	1.010	3.528	7.285

Table 4. Paired composition RMSE differences for RA-SMO minus MN-ASMO over 5–80 min. The mean difference and pointwise 95% confidence interval are in units of 10^{-3} mole fraction.

Scenario	Mean Δ	95% CI	Relative effect	MN-ASMO wins	Holmp-value
Nominal	-0.0152	[-0.0158, -0.0147]	-10.0%	0/30	5.57×10^{-31}
Noise	0.1360	[0.1192, 0.1528]	15.5%	30/30	7.47×10^{-16}
Mismatch	0.5237	[0.4429, 0.6045]	23.7%	30/30	2.34×10^{-13}
Combined	0.1893	[0.1411, 0.2375]	6.4%	30/30	2.23×10^{-8}

The estimated noise scale follows the imposed increases with a delay because it uses a trailing measurement window (Figure 4). Both MN-ASMO settings increase during these periods and decrease after the noise falls. The gain traces also differ between the two observers because they use different adaptation rates. The component study assesses the role of each noise-dependent update without changing those gain settings.

Effects of the two noise-dependent updates

Table 5 reports the matched component results, and Figure 5 shows paired effects under increased noise and combined uncertainty. Under increased noise, enabling both updates lowers mean composition RMSE by 53.5% relative to neither update, by 40.6% relative to Layer only, and by 10.7% relative to Threshold only. All 30 trials favor Both in each comparison, and all three Holm-adjusted p-values are below 10^{-20} . The threshold update provides the larger individual improvement, while adapting the layer gives a further benefit when the threshold already follows the noise estimate.

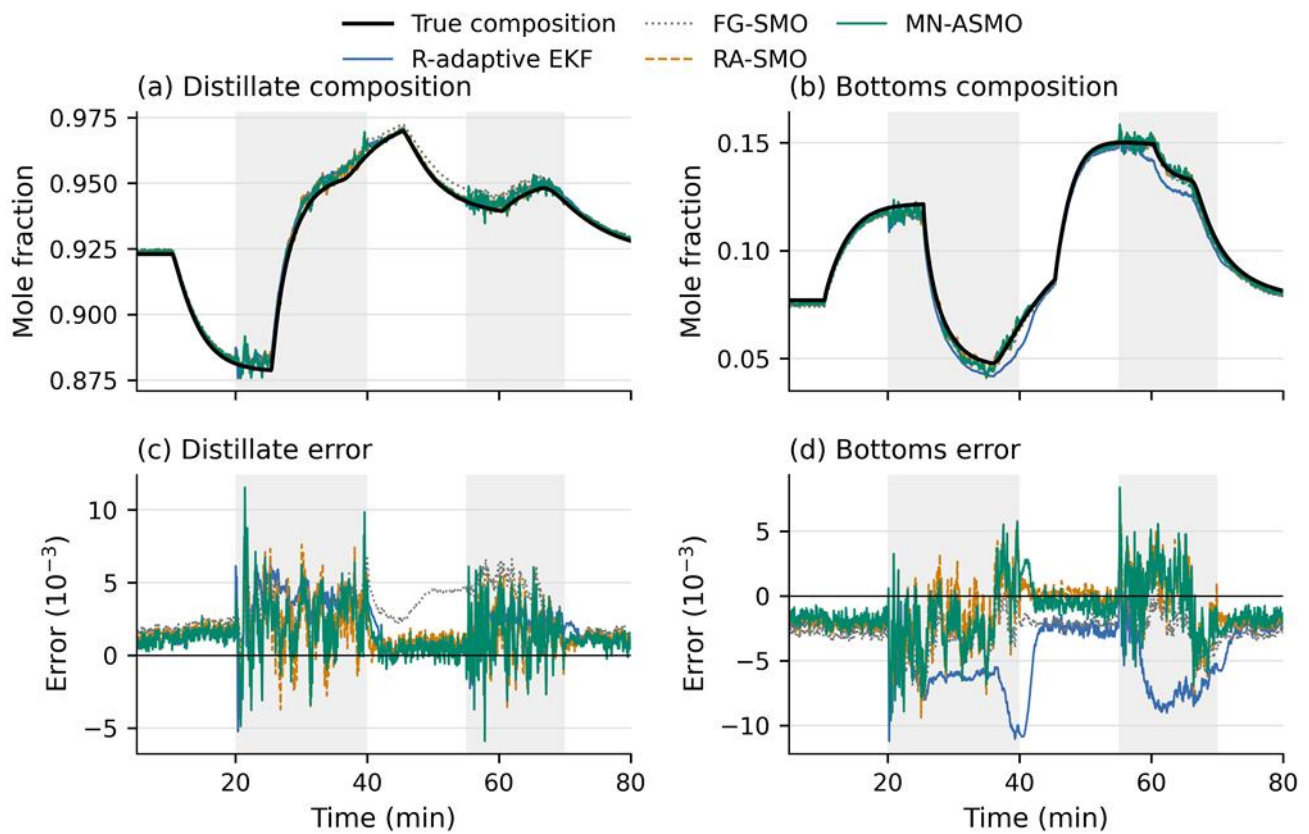


Figure 2. Estimated distillate and bottoms compositions and signed estimation errors for the first combined-uncertainty trial, seed 4001. Errors are estimated minus true composition. Gray shading marks the increased-noise periods.

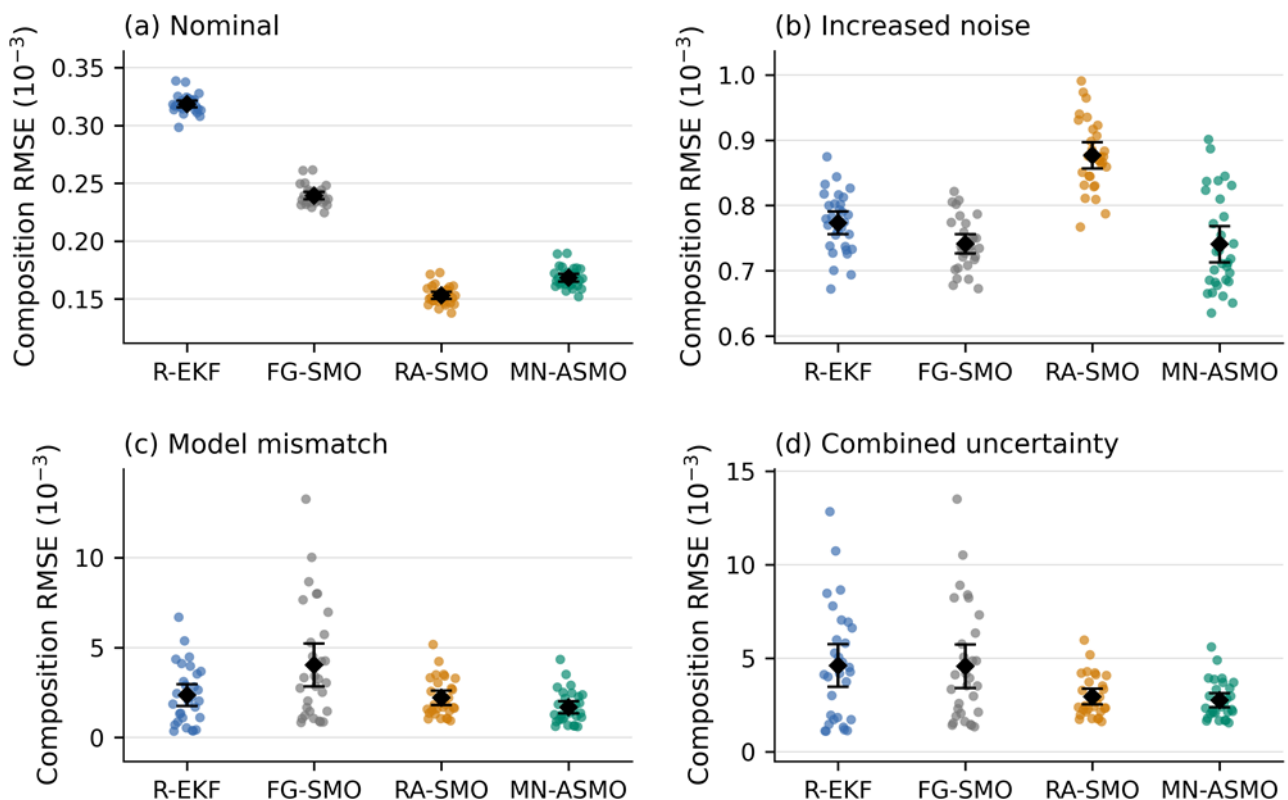


Figure 3. Composition RMSE over 5–80 min in 30 paired trials per scenario. Colored points show individual trials; black diamonds and bars show the mean and pointwise 95% confidence interval.

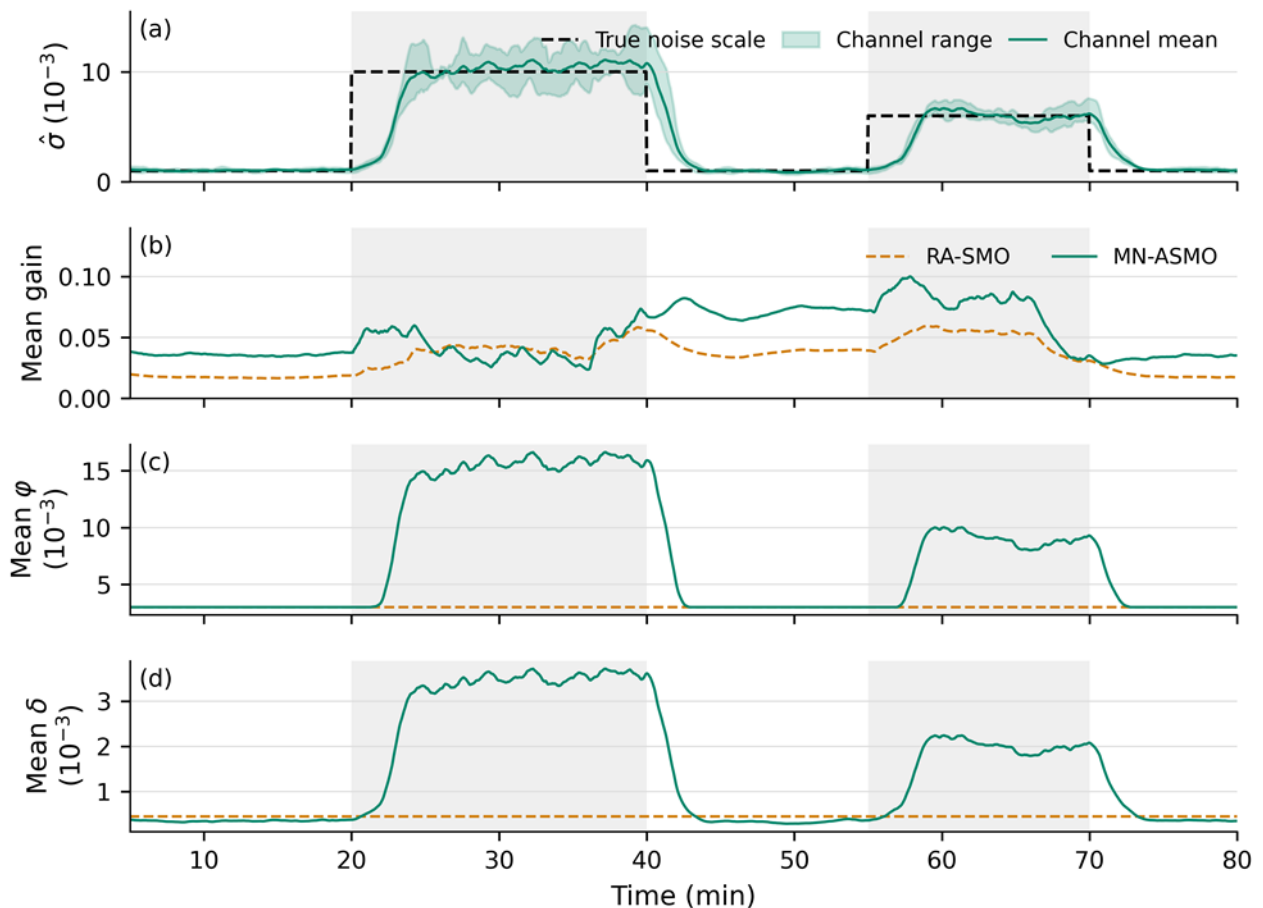


Figure 4. Noise estimates and adaptation variables for the same combined-uncertainty trial, seed 4001. Panel (a) shows the true noise standard deviation, the mean estimate, and the range across the four channels. Panels (b)–(d) show channel-mean gains, boundary-layer thicknesses, and dead-zone thresholds.

Table 5. Matched component study with $\gamma N = 30$ for every variant. Entries are means over 30 paired trials. RMSE is in units of 10^{-3} mole fraction; ErrorRateRMS is in units of 10^{-3} mole fraction/min. Both is the full MN-ASMO.

Scenario	Metric	Neither update	Layer only	Threshold only	Both
Nominal	RMSE	0.1675	0.1675	0.1684	0.1684
Nominal	ErrorRateRMS	1.8819	1.8819	1.8978	1.8978
Noise	RMSE	1.5941	1.2466	0.8292	0.7408
Noise	ErrorRateRMS	27.3270	19.3813	10.0674	8.7632
Mismatch	RMSE	1.6849	1.6849	1.6861	1.6861
Mismatch	ErrorRateRMS	9.7641	9.7641	9.7545	9.7545
Combined	RMSE	2.8795	2.7488	2.7473	2.7572
Combined	ErrorRateRMS	41.1895	33.4236	27.7188	24.5823

Under combined uncertainty, both lower the mean RMSE by 4.2% relative to neither update. The paired difference is 1.22×10^{-4} , with a pointwise 95% confidence interval of $[3.98 \times 10^{-5}, 2.05 \times 10^{-4}]$ and a Holm-adjusted p-value of 0.015. Its RMSE is slightly higher than that of either individual update, but both paired intervals include zero, and the adjusted p-value is 0.843. The joint update instead reduces ErrorRateRMS by 40.3%, 26.5%, and 11.3% relative to neither update, Layer only, and Threshold only, respectively. Under nominal conditions, Both has a small RMSE penalty of 0.56% relative to neither update. Under model mismatch, the corresponding difference is 0.07%, with a paired interval that includes zero. In these two scenarios, the boundary layer remains at its lower limit, so Layer coincides only with Neither, and Both coincides only with Threshold. The component study therefore does not support attributing the larger MN-ASMO versus RA-SMO mismatch improvement to noise adaptation.

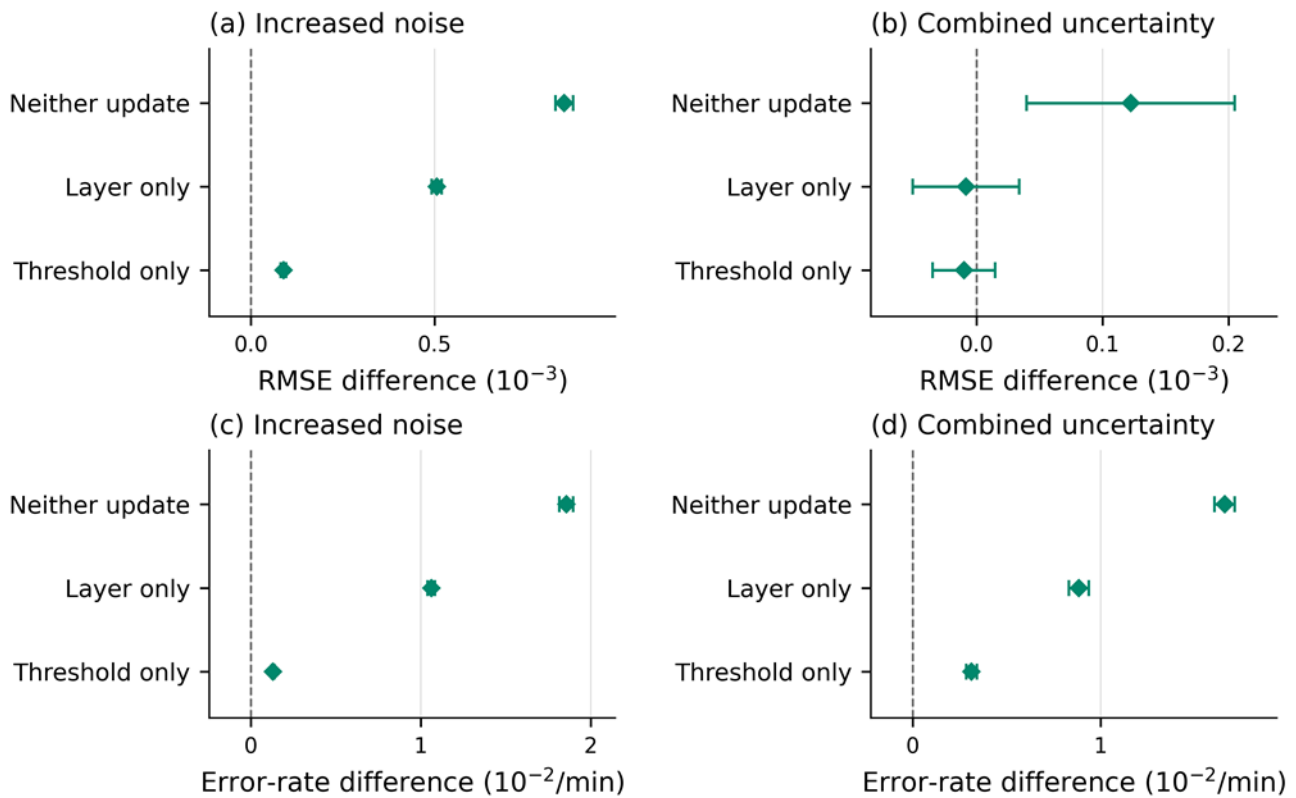


Figure 5. Matched component effects under increased noise and combined uncertainty. Diamonds and bars show the mean paired difference and pointwise 95% confidence interval. Panels (a) and (b) report composition RMSE; panels (c) and (d) report ErrorRateRMS. Horizontal scales differ across panels.

Error Dynamics and Computational Cost

Compared with RA-SMO, MN-ASMO reduces the mean ErrorRateRMS by 18.7% under increased noise but increases it by 24.7%, 79.0%, and 9.4% under nominal conditions, model mismatch, and combined uncertainty, respectively. Therefore, lower composition RMSE does not consistently indicate less temporal error variation. These configuration-level comparisons differ from the matched component results in Table 5.

Table 6. Computational profiling and implementation checks.

Item	R-Adaptive EKF	FG-SMO	RA-SMO	MN-ASMO
Mean step time (ms)	0.719	0.369	0.364	0.403
Worst P95 (ms)	1.979	0.879	0.832	0.914
Maximum (ms)	46.703	26.474	27.899	28.737
Deadline misses	0	0	0	0
Observer failures	0	0	0	0
State projections	0	0	0	0

Table 6 lists the computing time for the main comparison. MN-ASMO averages 0.403 ms per step, compared with 0.364 ms for RA-SMO, a 10.8% increase. Timing includes the causal noise estimate for MN-ASMO and the R-adaptive EKF. The longest recorded step among the main methods is 46.703 ms, below the 3000 ms sampling interval. We obtained these measurements in MATLAB R2021a on a 64-bit Windows platform. The archived results do not identify the processor, and desktop timing does not establish a hard real-time guarantee.

Finite-Horizon Observability

We assess local observability using the standard stacked observability matrix for a discrete-time linear model[25]. At each reflux ratio, we linearize the nominal column and calculate $F_d = \exp(AT_s)$. We stack $C, CF_d, \dots, CF_d^{400}$, giving $N = 401$ block rows over a 20 min horizon. This diagnostic uses the local linear model; the observer simulations use the full nonlinear RK4 model.

$$\mathcal{O}_N = \begin{bmatrix} C \\ CF_d \\ CF_d^2 \\ \vdots \\ CF_d^{N-1} \end{bmatrix}. \quad (46)$$

Its condition number is estimated from the singular values as

$$\kappa(\mathcal{O}_N) = \frac{\sigma_{\max}(\mathcal{O}_N)}{\sigma_{\min}(\mathcal{O}_N)}. \quad (47)$$

Table 7. Finite-horizon local observability diagnostics for the four-sensor configuration.

Reflux ratio	Rank at 10^{-6}	Rank at 10^{-8}	Rank at 10^{-10}	Condition number	Smallest singular value
2.5	29	32	32	3.610×10^7	7.232×10^{-8}
3.0	28	32	32	9.000×10^7	3.275×10^{-8}
3.5	28	30	32	3.198×10^8	1.173×10^{-8}

Figure 6 shows the rapid decay of the finite-horizon observability singular values, while the numerical values in Table 6 quantify the resulting conditioning. At sufficiently small singular-value thresholds, the finite-horizon matrix reaches numerical full rank, but the condition number is very large. It increases from 3.61×10^7 at $R=2.5$ to 3.20×10^8 at $R=3.5$, while the smallest singular value decreases. The practical message is that numerical rank alone is not enough: several state directions remain weakly recoverable from the four sensors.

Overall Discussion

The controlled component study conducted under increased noise conditions provides the strongest evidence supporting the proposed coupling. Employing both updates together yields a lower composition RMSE than using either update individually. The boundary-layer update decreases the correction slope, while the noise-dependent threshold limits gain increases in response to small residuals. Under combined uncertainty, the joint update reduces error variation but does not clearly improve RMSE relative to the individual updates. Therefore, the practical value of the joint update depends on whether the monitoring application prioritizes accuracy, error variation, or both. Comparison with RA-SMO reveals broader considerations regarding configuration choices. MN-ASMO demonstrates greater accuracy in the three non-nominal scenarios, whereas RA-SMO performs better under nominal conditions. The observed 23.7% reduction in mismatch is not attributable to the noise-dependent updates, as the component study shows no clear RMSE benefit from enabling these updates in that scenario. Alternative factors, including differences in gain rates and nominal thresholds, may account for the configuration-level performance differences. Additionally, MN-ASMO's accuracy under increased noise is comparable to FG-SMO, while the R-adaptive EKF achieves superior accuracy during noise-stress intervals. The conclusions rely on an idealized equilibrium-stage model that uses direct composition measurements. The simulations do not account for analyzer delay, sensor bias, energy dynamics, or temperature-to-composition conversion. Although parameter values may differ, the plant and observer models share the same structure. The noise estimation approach assumes locally smooth signals and approximately white measurement noise, with a delayed response to variance changes due to the trailing window. Furthermore, the local observability analysis indicates poorly conditioned directions.

Conclusion

This study evaluated the proposed MN-ASMO for estimating unmeasured distillate and bottoms compositions in a 32-state binary distillation column using four internal composition measurements. Across 30 paired Monte Carlo trials per scenario, MN-ASMO achieved mean composition RMSEs 15.5%, 23.7%, and 6.4% lower than RA-SMO under increased noise, model mismatch, and combined uncertainty, respectively. Its RMSE was 10.0% higher under nominal conditions. Under increased noise, its accuracy was similar to that of FG-SMO. The matched component study showed that updating the boundary-layer thickness and dead-zone threshold together reduced composition RMSE by 53.5% under increased measurement noise compared with keeping both settings fixed. The joint update also outperformed either update alone, with all other observer settings unchanged. Under combined uncertainty, it reduced error variation but offered no clear RMSE improvement over the individual updates. These results suggest that joint adaptation can improve estimation under increased measurement noise, although its benefits depend on the operating conditions. Future work should assess the effects of correlated noise, analyzer delay, and sensor bias, and validate the method using experimental column data.

Conflict of interest. Nil

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