

Original article

Stability and Continuous Dependence of the Solution of a Set-Valued Functional Equation with the Parameter

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Abstract

In this paper, we study the existence and stability of the Set-Valued functional equation with a parameter in $x \in L_1[0, T]$, $[1, 5, 12]$. The existence of solutions will be studied. The continuous dependence of the unique solution in $x \in L_1[0, T]$ will be studied. We study the uniqueness of the solution $x \in C[0, 1]$, and the continuous dependence of the unique solution will be studied. We study $x \in C[0, 1]$. We additionally provide examples given as the same application of the result.

Keywords: A Set-Valued Functional Equation: Lipschitz Condition: Banach Space.

Introduction

In this paper, we have many applications. Here, we are concerned with the set-valued equation with parameter

$$x(t) \in F(t, \lambda x(\phi(t)), \mu), \quad t \in [0, 1] \quad (1)$$

We study the existence of solutions $x \in L_1[0, 1]$ of (1). The continuous dependence of the unique solution will be studied [5, 3].

Main results

The following assumptions will be needed for our goals.

- (I) (i) The set $F(t, x, \mu)$ is nonempty, closed and convex for all $(t, x, \mu) \in [0, 1] \times R \times R$.
- (ii) $F(t, x, \mu)$ is satisfies continuous in x and μ for every $t \in [0, 1]$.
- (iii) There exists a bounded measurable function $a : [0, 1] \rightarrow R$ and a positive constant K , such that $\|F(t, x, \mu)\| = \sup \{ |f| : f \in F(t, x, \mu) \} \leq |a(t)| + K(|x| + |\mu|)$.

Remark 1

From the assumptions (i)-(iii) we can deduce that there exists $f \in F(t, x, \mu)$, such that

$$x(t) = f(t, x(\phi(t)), \mu), \quad \lambda > 0 \quad t \in (0, 1) \quad (2)$$

- (iv) $f : I \times R \times R \rightarrow R$ is measurable in t for any $x, \mu \in R$ and continuous in x, μ for $t \in [0, 1]$ such that there exists a bounded measurable function $a(t)$ and a positive constant $K > 0$ such that

$$|f(t, x(t), \mu)| \leq |a| + K(\|x\| + |\mu|), \quad \forall t \in I, \text{ and } x, \mu \in R \times R.$$

and

$$|a(t)| \leq a$$

- (II) $\phi : I \rightarrow I$ is absolutely continuous and $|\phi(t)| \leq t$, and $\exists B > 0$, such that, $\phi(s) > B$, (III) $K|\lambda| \leq K^*$ and, $a + K|\mu| \leq \mu^*$ and $\frac{K^*}{B} < 1$

Remark 2

From (I) and (iv), we can deduce that every solution of (2) is a solution of (1).

Existence of solutions

Theorem 2.1

Assume that the assumptions (I)-(III) are valid. Then the Problem (2) has at least one continuous solution $x \in L_1[0, T]$. proof. Let Q_r be a set defined as

$$Q_r = \{x \in L_1[0, 1] : \|x\| \leq r\}, \quad t \in [0, 1]$$

$$r = (\mu^*) \left(1 - \frac{K^*}{B}\right)^{-1}.$$

Define the operator A by $Ax(t) = f(t, \lambda x(\phi(t)), \mu)$ Let $x \in Q_r$; then

$$\int_T |Ax(t)| = \int_T |f(s, \lambda x(\phi(s)), \mu)| ds$$

$$\int_0^T |a(s)| ds + K(|\lambda| |x(\phi(t))| + |\mu|) ds$$

Taking $\phi(s) = u$ and $ds = \frac{du}{\phi'(s)}$, then

$$\leq a + K|\lambda| \int_{\phi(0)}^{\phi(T)} |x(u)| \frac{du}{\phi'(s)} + \int_0^1 K|\mu| ds$$

$$\leq a + \frac{K^*}{B} \|x\| + K|\mu|$$

$$\leq \mu^* + \frac{K^*}{B} r \leq r.$$

Hence, $Ax \in x$, which proves that $Ax \subset Q_r$ and $Ax \rightarrow Q_r$. This shows that the operator Ax maps L^1 into itself. Now, let us observe that the assumptions (I)-(II) imply that A is continuous on the set Q_r , and from assumption (I), f is continuous in x . Hence, we deduce that the operator A is continuous on Q_r . Finally, we will show that A is compact; to prove this, we will apply the Kolmogorov Compactness Criterion. Let Ω be a bounded subset of Q_r . Then $(A\Omega)$ is bounded in $L^1(I)$, i.e., condition (i) is satisfied. It remains to show that $(Ax)_h \rightarrow Ax$, as $h \rightarrow 0$ uniformly with respect to $Ax \in \Omega$. We have the following. Let $x \in \Omega \subset Q_r$, that is $x, \mu \in \Omega \subset Q_r$, then

$$\|(Ax)_h(t) - (Ax)(t)\| = \int_0^T \frac{1}{h} \int_t^{t+h} |Ax(\theta) - Ax(t)| d\theta ds$$

Since $Ax \in L^1$, then by the Lebesgue point Theorem.

$$\frac{1}{h} \int_t^{t+h} |Ax(\theta) - Ax(t)| d\theta \rightarrow 0$$

Hence, $\|(Ax)_h - (Ax)\| \rightarrow 0$, uniformly as $h \rightarrow 0$; then Ax is relatively compact, and there exists at least one fixed point $x \in L^1$ of (2)

Existence of A Unique Solution

Here we study the sufficient conditions for the uniqueness of the solution $x \in C[0,1]$ of problem (1). Consider the following assumptions

(I) (i) The set $F(t, x, \mu)$ is nonempty, compact and convex for all $F(t, x, \mu) \in [0, 1] \times R \times R$.

(ii) $F(t, x, \mu)$ is continuous and satisfies the Lipschitz condition with a positive constant K such that

$$H(F(t, x_1, \mu_1); F(t, x_2, \mu_2)) \leq K(\|x_1 - x_2\| + |\mu_1 - \mu_2|)$$

where $H(A, B)$ is the Hausdorff metric between the two subsets $A, B \subset I \times E$

Remark 2

From these assumptions, we can deduce that there exists a function $f \in F(t, x, \mu)$ such that

(iii) $F(t, x, \mu)$ is measurable in t for every $x \in R \times R$.

(iv) The set $f(t, x, \mu)$ is measurable in t for every $x, \mu \in R \times R$ and satisfies the Lipschitz condition with a positive constant K such that

$$|f(t, x_1, \mu_1) - f(t, x_2, \mu_2)| \leq K(\|x_1 - x_2\| + |\mu_1 - \mu_2|).$$

From this assumption (I), we have

$$|f(t, x, \mu) - f(t, 0, 0)| \leq |f(t, x, \mu) - f(t, 0, 0)| \leq K \|x\|.$$

Then

$$|f(t, x, \mu)| \leq K \|x\| + |f(t, 0, 0)| \leq K \|x\| + |a(t)|$$

where $|a(t)| = \sup_{t \in I} |f(t, 0, 0)|$, and $K|\lambda| \leq K^*$

(III) $\phi: I \rightarrow I$ is absolutely continuous and $|\phi(t)| \leq t$, and $\exists B > 0$, such that, $\phi'(s) > B$,

(III) $K|\lambda| \leq K^*$ and, $\frac{K^*}{B} < 1$

Theorem 3.1

Let the assumptions (I)-(III) be satisfied. Then the solution of the Problem (2) is unique.

Proof Let $x_1(t)$ and $x_2(t)$ be solution of the problem (2), then

$$\|x_1(t) - x_2(t)\| = \int_0^T (|f(s, \lambda x_1(\phi(s)), \mu) - f(s, \lambda x_2(\phi(s)), \mu)|) ds.$$

Taking $\phi(s) = u$ and $ds = \frac{du}{\phi'(s)}$, then

$$\leq K|\lambda| \int_{\phi(0)}^{\phi(T)} |x_1(u) - x_2(u)| \frac{du}{\phi'(s)}$$

$$\leq \frac{K^*}{B} \|x_1 - x_2\|$$

$$K^* \|x_1 - x_2\| \leq 0.$$

$$\text{Hence } \frac{K^*}{B} < 1,$$

$\|x_1 - x_2\| \leq (1 - \frac{K^*}{B})^{-1}$; then $x_1 = x_2$ and the solution of (2) is unique. This completes the proof.

Continuous Dependence of The Solution

Definition 1

The unique solution of the problem (2) depends continuously on λ if, for $\epsilon > 0, \exists \delta > 0$ such that

$$|\lambda - \lambda^*| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of

$$x^*(t) = f(t, \lambda x^*(\phi(t)), \mu) \quad (3)$$

Theorem 4.1

Let the assumption of Theorem 3.1 be satisfied; then the unique solution of (2) depends continuously on λ

Proof Let $x(t)$ and $x^*(t)$ be the solutions of problem (2); then

$$\|x(t) - x^*(t)\| = \int_0^T |(f(t, \lambda x(\phi(t)), \mu) - f(t, \lambda x^*(\phi(t)), \mu))| ds$$

$$\leq K \int_0^T |\lambda x(\phi(s)) - \lambda^* x^*(\phi(s))| ds$$

$$\leq K \int_0^T |\lambda x(\phi(s)) - \lambda x^*(\phi(s)) + \lambda x^*(\phi(s)) - \lambda^* x^*(\phi(s))| ds$$

Taking $\phi(s) = u$ and $ds = \frac{du}{\phi'(s)}$, then

$$\leq K \int_0^T (|\lambda x(\phi(s)) - \lambda x^*(\phi(s))| + |\lambda x^*(\phi(s)) - \lambda^* x^*(\phi(s))|) ds$$

$$\leq K(|\lambda| \int_{\phi(0)}^{\phi(T)} |x(u) - x^*(u)| \frac{du}{\phi'(s)} + \|\lambda - \lambda^*\| \int_{\phi(0)}^{\phi(T)} |x^*(u)| \frac{du}{\phi'(s)})$$

$$\leq \frac{K^*}{B} \|x - x^*\| + \frac{Kr\delta}{B} \leq r.$$

Then

$$(1 - K^*) \|x - x^*\| \leq (Kr\delta)$$

$$\|x - x^*\| \leq (\frac{Kr\delta}{1 - K^*})^{-1} \leq \epsilon$$

Definition 2

The unique solution of the problem (2) depends continuously on ϕ if $\epsilon > 0, \exists \delta > 0$ such that

$$|\phi(t) - \phi(t^*)| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of

$$x^*(t) = f(t, \lambda x^*(\phi^*(t)), \mu) \quad (4)$$

Theorem 4.2

Let the assumption of Theorem 3.1 be satisfied; then the unique solution of (2) depends continuously on λ

Proof: Let $x(t)$ and $x^*(t)$ be the solutions of problem (2), then

$$\|x(t) - x^*(t)\| = \int_0^T |f(s, \lambda x(\phi(s)), \mu) - f(s, \lambda x^*(\phi^*(s)), \mu)| ds$$

$$\leq K \int_0^T |\lambda x(\phi(s)) - \lambda x^*(\phi(s)) + \lambda x^*(\phi(s)) - \lambda x^*(\phi^*(s))| ds$$

Taking $\phi(s) = u$ and $ds = \frac{du}{\phi'(s)}$, then

$$\leq K|\lambda| \int_{\phi(0)}^{\phi(T)} |x(u) - x^*(u)| \frac{du}{\phi'(s)} + K|\lambda| |\phi - \phi^*| \int_{\phi(0)}^{\phi(T)} |x^*(u)| \frac{du}{\phi'(s)}$$

$$\leq \frac{K^*}{B} \|x - x^*\| + \frac{K^*r\delta}{B} \leq \epsilon.$$

Then

$$\frac{(1 - K^*)}{B} \|x - x^*\| \leq \left(\frac{K^* r \delta}{B}\right)$$

$$\|x - x^*\| \leq \left(\frac{K^* r \delta}{B}\right) \left(1 - \frac{K^*}{B}\right)^{-1} \leq \epsilon$$

Definition 3

The unique solution of the problem (2) depends continuously on μ if $\epsilon > 0, \exists \delta > 0$ such that

$$|\mu - \mu^*| \leq \delta \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of

$$x^*(t) = f(t, \lambda x^*(\phi(t)), \mu^*) \quad (5)$$

Theorem 4.3

Let the assumption of Theorem 3.1 be satisfied; then the unique solution of (2) depends continuously on λ

Proof. Let $x(t)$ and $x^*(t)$ be the solutions of problem (2).

$$\|x(t) - x^*(t)\| = \int_0^T |f(s, \lambda x(\phi(s)), \mu) - f(s, \lambda x^*(\phi(s)), \mu^*)| ds$$

Taking $\phi(s) = u$ and $ds = \frac{du}{\phi'(s)}$, then

$$\leq K|\lambda| \int_{\phi(0)}^{\phi(T)} |x(u) - x^*(u)| \frac{du}{\phi'(s)} + K \int_0^T |\mu - \mu^*| ds$$

$$\frac{K|\lambda|}{B} \|x - x^*\| + \frac{K\delta}{B}$$

$$\frac{K^*}{B} \|x - x^*\| + \frac{K\delta}{B} \epsilon.$$

Then

$$\left(1 - \frac{K^*}{B}\right) \|x - x^*\| \leq \frac{(K\delta)}{B}$$

$$\|x - x^*\| \leq \left(\frac{K\delta}{B}\right) \left(1 - \frac{K^*}{B}\right)^{-1} \leq \epsilon$$

Example 1

Consider the following equation.

$$x(t) = t^4 e^{-t} + \lambda x(\cos 4t + 1) + \frac{1}{2}$$

Then

$$|f(s, \lambda x(\phi(s)), \mu)| = |t^4 e^{-t} + \lambda x(\cos(4t + 1)) + \frac{1}{4}|$$

$$\leq \frac{1}{2} (|\lambda| |x(\cos(4t + 1))| + \left|\frac{1}{2}\right|)$$

It is clear that the assumptions (I) and (III) of Theorem 2.1 are satisfied with $a(t) = t^4 \exp(-t) \in L_1[0, T]$, and $\phi(t) = \cos(4t + 1)$, and let, $\lambda = \frac{1}{2}, k = \frac{1}{2}$, then $k^* = k\lambda = \frac{1}{4} < 1$, Therefore, by applying Theorem 3.1, the Set-valued equation (1) has a continuous solution.

Example 2

Consider the following equation; since equation (3) is transformed to function equation (5), then

$$\frac{dx(t)}{dt} = \frac{1}{2}(t - 3) + \lambda(x(\sin(5t))) + \frac{1}{2}\mu,$$

then

$$|f(t, \frac{1}{2}(t - 3) + \lambda(x(\sin(5t))) + \frac{1}{2}\mu, |$$

$$\leq \left|\frac{1}{2}(t - 3)\right| + |\lambda(x(\sin(5t)))| + \left|\frac{1}{2}\mu\right|$$

It is clear that the assumptions (1) and (2) of Theorem 3.1 are satisfied with $a(t) = \frac{1}{2}(t - 3) \in C(0, T]$, and let, $\lambda = 1, L = \frac{1}{2}$, and $\phi(t) = \sin(5t)$; therefore, by applying Theorem 3.1, the equation (3) has a continuous solution.

References

1. Agarwal R, Meehan M, O'Regan D. Fixed point theory and applications. Cambridge: Cambridge University Press; 2001. (Cambridge Tracts in Mathematics).
2. El-Sayed AMA, Ba-Ali MMS, Hamdallah EMA. An investigation of a nonlinear delay functionalequation with a quadratic functional integral constraint. *Mathematics*. 2023;11:4475. doi:10.3390/math11214475.
3. Foulkes JB. Set-valued functions and convexity; 1980 May.
4. Borisovich YuG, Gel'man BD, Myshkis AD, Obukhovskii VV. Multivalued mappings. *Uspekhi Mat Nauk*. 1980;35(1):59-126.
5. Cardinali T, Nikodem K, Papalini F. Some results on stability and characterization of K-convexity of set-valued functions. *Ann Polon Math*. 1993;58:185-192.
6. Cichon M, El-Sayed AMA, Hussien A. Existence theorems for nonlinear functional integral equations of fractional orders. *Commentat Math*. 2001;41:59-67.
7. Coban MM. Multivalued mappings and Borel sets, 1. *Trudy Mosk Mat Obsc*. 1970;22:229-250.
8. Coban MM. Multivalued mappings and Borel sets, 2. *Trudy Mosk Mat Obsc*. 1970;23:277-301.
9. Kikuchi N. On contingent equations satisfying the Caratheodory type conditions. *Publ Res Inst Math Sci Kyoto Univ Ser A*. 1968;3:361-371.
10. Kupka I. Continuous selection for Lipschitz multifunctions. *Acta Math Univ Comenianae*. 2005;74(1):133-141.
11. Nikodem K. On Jensen's functional equation for set-valued functions. *Rad Math*. 1987;3:23-33.
12. Nikodem K. Set-valued solutions of the Pexider functional equations. *Funkcial Ekvac*. 1988;31:227-231.
13. Park S. Applications of Michael's selection theorems to fixed point theory. *Topology Appl*. 2008;155:861-870.
14. van Tiel J. Convex analysis: an introductory text. De Bilt: Royal Netherlands Meteorological Institute; 1984.