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Original article

A Novel Asymmetric Fuzzy Number with Damped Resonance: The AFRN Framework

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Abstract
Classical fuzzy numbers are widely used for representing uncertainty and imprecision, but their conventional convex structure may be insufficient for uncertainty profiles exhibiting local oscillations or resonance-like behaviour. This paper introduces the Asymmetric Fuzzy Resonance Number (AFRN), a six-parameter fuzzy model constructed from an asymmetric Laplacian kernel modulated by a damped cosine term. The proposed membership function preserves normality at its core while allowing damped oscillatory behaviour away from the centre. We establish its basic analytical properties, including normality, boundedness, continuity, non-differentiability at the core, and recovery of the classical asymmetric Laplacian in the appropriate limiting cases. Furthermore, a closed-form expression for the Centre of Gravity (COG) defuzzification is derived, avoiding direct numerical integration. Alpha-cuts are also characterised in terms of their possible connected components. Finally, a parametric sensitivity study illustrates the effects of the resonance amplitude and frequency on the membership profile and the defuzzified value. The AFRN provides a flexible parametric framework for representing asymmetric fuzzy uncertainty profiles with damped local oscillations.
Keywords. Fuzzy Number, Asymmetric Membership Function, Damped Resonance, Alpha-Cuts, Defuzzification.

Introduction

Since the seminal work of Zadeh [1], fuzzy set theory has become an important framework for modelling uncertainty and imprecision in complex systems. Fuzzy numbers, commonly defined as normalised and convex fuzzy sets on the real line, have been extensively applied in control, decision-making, optimisation, and pattern recognition [2-3]. Classical fuzzy numbers, including triangular, trapezoidal, and many Laplacian-type models, generally employ membership functions with a single-peaked and convex structure. Consequently, their alpha-cuts are single closed intervals [2-3]. Although this structure provides important mathematical advantages, it may be restrictive when the uncertainty profile contains local fluctuations or oscillatory behaviour.

To represent asymmetrical uncertainty, several asymmetric membership functions have been proposed. For example, asymmetric membership functions have been considered in neural fuzzy systems [4], while asymmetric Laplace-type fuzzy information granules have been investigated for time-series applications [5]. These approaches provide useful control over left and right spreads, but they do not specifically incorporate a damped oscillatory component. Non-convex fuzzy sets have also been investigated in the fuzzy-set literature. Chakraborty [6] studied canonical representations of discrete fuzzy numbers, while other studies have considered the construction and application of non-convex fuzzy sets [7-8]. These developments indicate that non-convex membership structures can be useful when conventional convex representations are insufficient. Motivated by these considerations, this paper introduces the Asymmetric Fuzzy Resonance Number (AFRN). The model combines an asymmetric Laplacian kernel with a damped cosine modulation. The factor $\cos(\omega(x-c))-1$ is used so that the oscillatory perturbation vanishes exactly at the core $x=c$, thereby preserving the normalisation condition $\mu(c)=1$. The proposed framework is characterised by an asymmetric membership function that incorporates a damped oscillatory component governed by explicit amplitude, frequency, and damping parameters. The analytical properties of the model are rigorously established, including normality, boundedness, continuity, and the presence of non-differentiability at the core. Furthermore, the framework generalises the classical asymmetric Laplacian membership function, which is recovered as a particular case when the resonance amplitude is set to zero.

A closed-form expression for Centre of Gravity (COG) defuzzification is derived, providing an efficient analytical tool for inference. In addition, the alpha-cut structure is investigated, revealing that the corresponding level sets may be either connected or disconnected depending on the selected parameter values. To illustrate the practical impact of the proposed formulation, a numerical sensitivity analysis is conducted to examine the influence of the resonance parameters on both the shape of the membership function and the resulting COG.

The remainder of the paper is organised as follows. Section 2 introduces the mathematical definition of the AFRN. Section 3 establishes its analytical properties. Section 4 presents the closed-form COG expression and discusses the alpha-cuts. Section 5 provides the numerical investigation and parameter sensitivity analysis. Section 6 concludes the paper.

Mathematical Definition of the AFRN

Let $x\in\mathbb{R}$ be a physical variable and let $c\in\mathbb{R}$ denote the centre of the fuzzy model. The proposed AFRN is defined by a membership function with asymmetric left and right spreads.

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Definition: AFRN Membership Function

The membership function of the Asymmetric Fuzzy Resonance Number (AFRN) is defined by

$$\mu(x) = \begin{cases} \exp\left(-\frac{|x-c|}{\sigma(x)}\right) \cdot \left(1 + \varepsilon\left[\cos\left(\omega(x-c)\right) - 1\right]. \exp\left(-\frac{|x-c|}{\tau}\right)\right), & x \in \mathbb{R} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where

$$\sigma(x) = \begin{cases} \sigma_L, & x \leq c, \\ \sigma_R, & x > c. \end{cases} \tag{2}$$

The parameters satisfy the regularity conditions

$$\sigma_L > 0, \qquad \sigma_R > 0, \qquad \tau > 0, \qquad 0 \leq \varepsilon \leq \tfrac{1}{2}, \qquad \omega > 0. \tag{3}$$

Here, c controls the centre, σ_L and σ_R determine the left and right spreads, respectively, ε controls the amplitude of the resonance term, ω controls its frequency, and τ controls the spatial damping. The factor $\cos(\omega(x-c)) - 1$ vanishes at $x = c$. Consequently, the resonance perturbation does not change the membership value at the core. The model therefore contains six parameters: $(c, \sigma_L, \sigma_R, \varepsilon, \omega, \tau)$.

Analytical Properties

This section establishes the fundamental properties of the AFRN membership function.

Theorem 3.1 Normality and Boundedness

The membership function defined by (1) is normal and bounded. Specifically,

$$\mu(c) = 1, \tag{4}$$

and

$$0 \leq \mu(x) \leq 1, \; x \in \mathbb{R}. \tag{5}$$

Furthermore,

$$\lim_{x \to \pm\infty} \mu(x) = 0. \tag{6}$$

Proof. At $x = c$, we have

$$\exp(0) = 1, \qquad \cos(0) - 1 = 0,$$

and therefore

$$\mu(c) = 1.$$

Moreover,

$$-2 \leq \cos(\omega(x-c)) - 1 \leq 0.$$

Since

$$0 \leq \varepsilon \leq \frac{1}{2}$$

and

$$0 < exp\left(-\frac{|x-c|}{\tau}\right) \leq 1,$$

it follows that

$$-1 \leq \varepsilon\left[\cos\left(\omega(x-c)\right) - 1\right]\exp\left(-\frac{|x-c|}{\tau}\right) \leq 0.$$

Hence,

$$0 \leq 1 + \varepsilon\left[\cos\left(\omega(x-c)\right) - 1\right]\exp\left(-\frac{|x-c|}{\tau}\right) \leq 1.$$

Since

$$0 < exp\left(-\frac{|x-c|}{\sigma(x)}\right) \leq 1,$$

we obtain

$$0 \leq \mu(x) \leq 1.$$

Finally, because

$$\sigma_L > 0, \qquad \sigma_R > 0,$$

the leading exponential factor satisfies

$$\exp\left(-\frac{|x-c|}{\sigma(x)}\right) \to 0 \qquad \text{as } x \to \pm\infty.$$

The second factor is bounded. Therefore,

$$\lim_{x \to \pm\infty} \mu(x) = 0. \square$$

Theorem 3.2 Continuity and Differentiability

The AFRN membership function $\mu(x)$ is continuous on $\mathbb{R}$. However, it is not differentiable at $x = c$ for all admissible $\sigma_L, \sigma_R > 0$.

Proof. For $x \neq c$, $\mu(x)$ is a product of continuous elementary functions on each side of c. It remains to examine the behaviour at the core.

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Let

$$h = x - c,$$

so that

$$x = c + h.$$

As $h \to 0$, we have the Taylor expansions

$$\cos(\omega h) - 1 = -\frac{\omega^2 h^2}{2} + O(h^4),$$

and

$$\exp\left(-\frac{|h|}{\tau}\right) = 1 + O(|h|).$$

Therefore, the resonance contribution satisfies

$$\varepsilon(\cos(\omega h) - 1)\exp\left(-\frac{|h|}{\tau}\right) = \varepsilon\left(-\frac{\omega^2 h^2}{2} + O(h^4)\right)\big(1 + O(|h|)\big) = O(h^2),$$

since $h^2 . 1 = O(h^2)$ and all higher-order terms are absorbed into $O(h^2)$. Hence, the resonance term is of order $O(h^2)$ near the core and does not contribute to the first-order derivative.

Left derivative. For $h < 0$, we have $|h| = -h$ and $\sigma(c + h) = \sigma_L$. Therefore,

$$\mu(c + h) = \exp\left(-\frac{|h|}{\sigma_L}\right)[1 + O(h^2)] = \exp\left(\frac{h}{\sigma_L}\right)[1 + O(h^2)].$$

Using $\exp\left(\frac{h}{\sigma_L}\right) = 1 + \frac{h}{\sigma_L} + O(h^2)$, we obtain

$$\mu(c + h) = \left(1 + \frac{h}{\sigma_L} + O(h^2)\right)[1 + O(h^2)] = 1 + \frac{h}{\sigma_L} + O(h^2).$$

Thus,

$$\mu'_-(c) = \lim_{h\to 0^-}\frac{\mu(c + h) - \mu(c)}{h} = \lim_{h\to 0^-}\frac{\frac{h}{\sigma_L} + O(h^2)}{h} = \frac{1}{\sigma_L}.$$

Right derivative. For $h > 0$, we have $|h| = h$ and $\sigma(c + h) = \sigma_R$. Therefore,

$$\mu(c + h) = \exp\left(-\frac{h}{\sigma_R}\right)[1 + O(h^2)] = \left(1 - \frac{h}{\sigma_R} + O(h^2)\right)[1 + O(h^2)] = 1 - \frac{h}{\sigma_R} + O(h^2),$$

Thus,

$$\mu'_+(c) = \lim_{h\to 0^+}\frac{\mu(c + h) - \mu(c)}{h} = \lim_{h\to 0^+}\frac{-\frac{h}{\sigma_R} + O(h^2)}{h} = -\frac{1}{\sigma_R}.$$

Since $\sigma_L > 0$ and $\sigma_R > 0$, we have $\mu'_-(c) > 0$ and $\mu'_+(c) < 0$. Therefore, the one-side derivatives cannot be equal, and μ is not differentiable at $x = c$.

Continuity at c follows directly from

$$\lim_{x\to c}\mu(x) = \mu(c) = 1.$$

Hence, μ is continuous on $\mathbb{R}$ but not differentiable at the core. □

Remark Conditional Non-Convexity

The presence of the damped cosine term allows the AFRN membership function to exhibit local oscillations and, for suitable combinations of ε, ω, τ, σ_L, and σ_R, non-convex profiles can occur. Thus, non-convexity is parameter-dependent and should not be asserted for every $\varepsilon \neq 0$. In particular, the AFRN framework includes both regimes in which the membership profile remains compatible with connected alpha-cuts and regimes in which disconnected alpha-cuts may occur.

Theorem 3.3 Recovery of the Classical Asymmetric Laplacian

When the resonance amplitude is deactivated, i.e., $\varepsilon = 0$, the AFRN reduces to the classical asymmetric Laplacian membership function:

$$\mu_0(x) = \exp\left(-\frac{|x-c|}{\sigma(x)}\right). \quad (7)$$

Moreover, for $x \neq c$,

$$\lim_{\tau\to 0^+}\mu(x) = \exp\left(-\frac{|x-c|}{\sigma(x)}\right). \quad (8)$$

Proof. Setting $\varepsilon = 0$ in (1) gives

$$\mu(x) = \exp\left(-\frac{|x - c|}{\sigma(x)}\right),$$

which is exactly the classical asymmetric Laplacian form.

For $x \neq c$ and $\tau \to 0^+$,

$$\exp\left(-\frac{|x - c|}{\tau}\right) \to 0.$$

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Hence the resonance contribution vanishes and

$$\lim_{\tau\to 0^+} \mu(x) = \exp\left(-\frac{|x-c|}{\sigma(x)}\right).$$

At $x = c$, the resonance contribution is zero for every $\tau > 0$, and $\mu(c) = 1$ remains unchanged. □

Defuzzification and Alpha-Cuts

Closed-Form Centre of Gravity

The Centre of Gravity (COG) is a widely used defuzzification method for mapping a fuzzy output to a representative crisp value [9].

$$x^* = \frac{\int_{-\infty}^{\infty} x\mu(x)\,dx}{\int_{-\infty}^{\infty} \mu(x)\,dx} = \frac{N}{D}. \quad (9)$$

To derive a closed-form expression, introduce

$$a_L = \frac{1}{\sigma_L} + \frac{1}{\tau}, \qquad a_R = \frac{1}{\sigma_R} + \frac{1}{\tau}. \quad (10)$$

We use the standard Laplace-transform identity

$$\int_0^\infty e^{-ay}\cos(\omega y)\,dy = \frac{a}{a^2+\omega^2},\ \ a > 0. \quad (11)$$

The denominator of (9) is

$$D = (\sigma_L + \sigma_R) - \varepsilon\omega^2\left[\frac{1}{a_L(a_L^2+\omega^2)} + \frac{1}{a_R(a_R^2+\omega^2)}\right]. \quad (12)$$

Define the auxiliary function.

$$I(a) = -\frac{\omega^2(3a^2+\omega^2)}{a^2(a^2+\omega^2)^2}. \quad (13)$$

Then the numerator shift relative to the centre is

$$N - cD = (\sigma_R^2 - \sigma_L^2) + \varepsilon[I(a_R) - I(a_L)]. \quad (14)$$

Consequently, the exact COG value is

$$x^* = c + \frac{(\sigma_R^2-\sigma_L^2)+\varepsilon[I(a_R)-I(a_L)]}{(\sigma_L+\sigma_R)-\varepsilon\omega^2\left[\frac{1}{a_L(a_L^2+\omega^2)}+\frac{1}{a_R(a_R^2+\omega^2)}\right]}. \quad (15)$$

When $\varepsilon = 0$, the resonance term disappears and

$$x^* = c + \frac{\sigma_R^2-\sigma_L^2}{\sigma_L+\sigma_R} = c + \sigma_R - \sigma_L. \quad (16)$$

Thus, the classical asymmetric centre is recovered exactly.

Alpha-Cuts

For $0 < \alpha \leq 1$, the α-cut of the AFRN is defined by [2-3].

$$A_\alpha = \{x \in \mathbb{R} : \mu(x) \geq \alpha\}. \quad (17)$$

Depending on the parameter regime, the α-cut may be a single interval,

$$A_\alpha = [x_L(\alpha), x_R(\alpha)], \quad (18)$$

or may consist of several connected components:

$$A_\alpha = \bigcup_{i=1}^{n(\alpha)} [x_{L,i}(\alpha), x_{R,i}(\alpha)]. \quad (19)$$

The boundary points satisfy

$$\mu(x) = \alpha. \quad (20)$$

Because of the oscillatory term, these equations generally do not possess a simple closed-form solution. The boundary points can therefore be computed numerically, for example by applying a bracketing method such as bisection on intervals containing sign changes of $\mu(x) - \alpha$. The number of connected components $n(\alpha)$ depends on the complete set of AFRN parameters and on the selected α-level. Consequently, no universal monotonic relationship between $n(\alpha)$ and either ω or α is assumed.

Numerical Investigation and Parametric Sensitivity

To illustrate the behaviour of the AFRN, we consider the baseline parameter set

$$c = 0, \qquad \sigma_L = 1, \qquad \sigma_R = 2, \qquad \tau = 2. \quad (21)$$

The resonance amplitude ϵ and frequency ω are varied while the remaining parameters are kept fixed.

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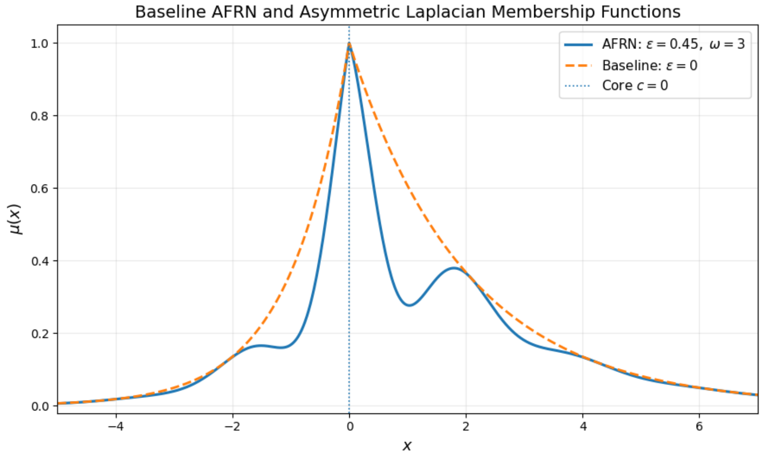

Figure 1. Baseline AFRN and Asymmetric Laplacian Membership Functions

Baseline AFRN membership profile for $c = 0$, $\sigma_L = 1$, $\sigma_R = 2$, $\tau = 2$, with resonance parameters selected to illustrate the damped oscillatory behaviour, together with the corresponding asymmetric Laplacian baseline.

Figure 1 illustrates how the damped resonance term modifies the asymmetric Laplacian profile while preserving the membership value $\mu(c) = 1$.

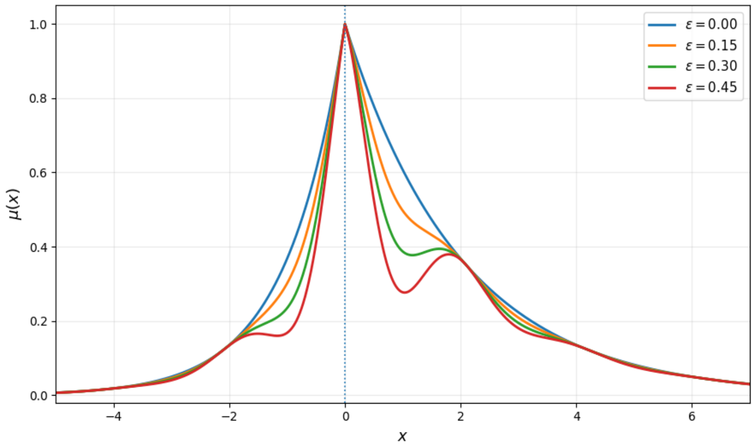

Figure 2. Sensitivity of AFRN Membership Function to ε.

Effect of the resonance amplitude ε on the AFRN membership function for $\omega = 3$, with $c = 0$, $\sigma_L = 1$, $\sigma_R = 2$, and $\tau = 2$. As ε increases within the admissible range, the oscillatory component becomes more pronounced. The effect remains damped as $|x - c|$ increases because of the exponential factor

$$\exp\left(-\frac{|x - c|}{\tau}\right).$$

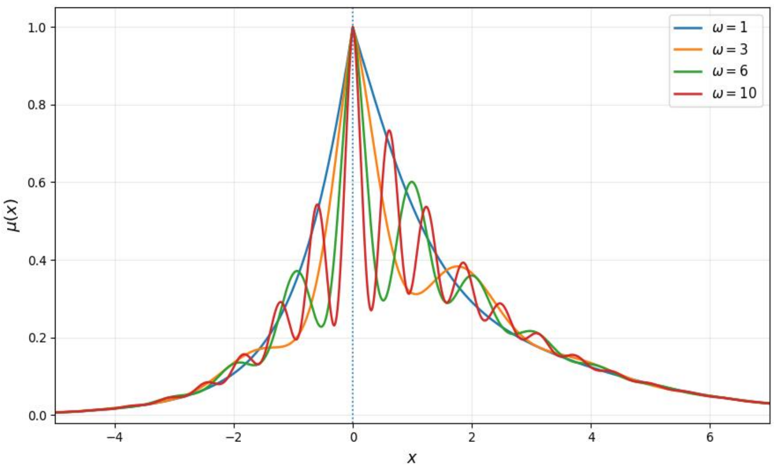

Figure 3. Sensitivity of AFRN Membership Function to ω.

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Effect of the resonance frequency ω on the AFRN membership function for $\varepsilon = 0.40$, with $c = 0$, $\sigma_L = 1$, $\sigma_R = 2$, and $\tau = 2$. Figure 3 shows that increasing ω compresses the oscillatory structure spatially, producing more rapid local fluctuations while the overall oscillatory contribution remains controlled by the damping parameter τ.

Sensitivity of the Centre of Gravity

Using the closed-form expression (15), we evaluate the COG for several values of ε and ω. The baseline parameters are

$$\sigma_L = 1, \qquad \sigma_R = 2, \qquad \tau = 2, \qquad c = 0.$$

The resulting values are presented in Table 1.

Centre of Gravity x^* for varying resonance amplitude ε and frequency ω. Baseline: $c = 0$, $\sigma_L = 1$, $\sigma_R = 2$, $\tau = 2$.

Table 1. AFRN COG values

ε	ω			
	1	3	6	10
0.00	1.0000	1.0000	1.0000	1.0000
0.15	1.0016	1.0458	1.0567	1.0592
0.30	1.0034	1.0994	1.1241	1.1300
0.45	1.0052	1.1626	1.2057	1.2162

At $\varepsilon = 0$, the value $x^* = 1$ is recovered exactly, which agrees with $c + \sigma_R - \sigma_L = 0 + 2 - 1 = 1$. For the parameter grid considered in Table 1, increasing ε shifts the COG toward the wider right side. The magnitude of this shift depends on the frequency ω. Similarly, increasing ω produces an increase in the COG for the selected parameter values, with a progressively smaller change between the largest frequencies in the table. These observations describe the selected numerical parameter grid and are not intended as universal monotonicity claims for all admissible parameter combinations.

Conclusion and Future Work

This paper introduced the Asymmetric Fuzzy Resonance Number (AFRN), a six-parameter fuzzy model combining an asymmetric Laplacian kernel with a damped cosine modulation. The proposed formulation preserves normality at the centre and provides explicit control over asymmetry, resonance amplitude, oscillation frequency, and damping. The basic analytical properties of the membership function were established. In particular, the membership function is normal, bounded, and continuous, while it is non-differentiable at the core because the left and right one-sided derivatives have opposite signs. The classical asymmetric Laplacian membership function is recovered when the resonance amplitude is set to zero.

A closed-form expression for the Centre of Gravity was also obtained. This expression provides an analytical alternative to direct numerical integration and recovers the classical asymmetric centre in the absence of resonance. The analysis of alpha-cuts shows that their connectedness depends on the parameter regime. In particular, the damped oscillatory component can generate local fluctuations that may lead to disconnected alpha-cuts for appropriate parameter values. Thus, the AFRN framework can represent uncertainty profiles that are not restricted to a single convex shape. The numerical investigation illustrated the effects of the resonance amplitude and frequency on the membership profile and the defuzzified output. For the parameter grid considered, increasing the resonance amplitude and frequency modified the COG toward the wider side of the asymmetric membership profile.

Future work will investigate the application of the AFRN framework in uncertainty-aware neuro-fuzzy systems, signal processing, and other applications in which asymmetric and locally oscillatory uncertainty profiles arise. Further theoretical investigation of sufficient conditions for non-convexity and of higher-dimensional extensions is also possible.

Conflict of interest. Nil

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