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Original article

Analytical Methods for Fractional Differential Equation Systems Using Real Symmetric Matrices

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Abstract
This work aims to study the methods for finding exact solutions of fractional differential equation systems involving real symmetric matrices, homogeneous and nonhomogeneous, by using the Laplace transform and Mittag-Leffler function methods, with some illustrative examples.
Keywords. Gamma Function, Riemann-Liouville and Caputo Fractional, Laplace Transform, Mittag-Leffler Function, System of Fractional Equation Matrix

Introduction

The idea it is well known that fractional differential equations are generalisations of ordinary differential equations to an arbitrary non- integer order. In the theory of integer-order differential equations, the exponential function, i.e. e^z plays a paramount role. For example, to find the solution of $y' + y = 0$. In a like manner, the role of the Mittag-Leffler function is very critical for fractional order calculus, and it also plays a pivotal role in finding the solution of non-integer-order differential equations [9]. The Mittag-Leffler function was introduced in 1903 A.D. in multiple fields of mathematics and physics, including fractional calculus and the theory of fractional differential equations. It is inseparably connected to fractional calculus, which extends the classical calculus to non- integer orders [3,6, 7, 8, 12]. This function is special in mathematics and generalises the exponential function to fractional and complex arguments [6]. A two-parameter function is defined as,

$$E_{r,s}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(rk+s)'} \quad r,s > 0 \;, r,s \in \mathbb{R} \;, z \in \mathbb{C}$$

Some fractional differential equations are solved by using the methods of the fractional Laplace transform and the Mittag-Leffler function. In this paper, we explore. In section 2, we define the Gamma function; we also define the Laplace transform for Riemann-Liouville and Caputo fractional derivatives, and define the Mittag-Leffler function with relations and examples. In section 3, we focus on the system of fractional equations and real symmetric matrices, and also verify the results with solutions of homogeneous and non–homogeneous systems of examples.

Preliminaries

In this section, we recall some definitions which would be useful for subsequent discussions. First, we introduce the definition of the Gamma function, which is simply the generalisation of the factorial for all real numbers.

Definition Gamma Function [10]

The Gamma function, denoted by $\Gamma(z)$, is defined on the complex plane, i.e

$$\Gamma(\mathrm{z}) = \int_0^{\infty} e^{-t}\; t^{z-1}\mathrm{dt}\,, \qquad (\mathrm{Re\,z} > 0) \tag{1}$$

With the relations: $\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(\mathrm{z})$, and $\Gamma(\mathrm{z}) = (\mathrm{z}-1)!, \mathrm{z} \in \mathbb{N}$
The Gamma function is fundamental in fractional calculus as it generalises the fractional function to non- integer order, providing the necessary framework for defining fractional derivatives.

Definition: Riemann-Liouville's Fractional Derivative [2]

Suppose that $n \in \mathbb{N}$. The Riemann-Liouville fractional derivative ${}^{R}D_t^{\alpha}$ of order $\alpha \in (n-1, n]$ for a function v is defined as

$${}^{R}D_t^{\alpha} v(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\zeta)^{n-\alpha-1} v(\zeta) d\zeta \,, t > 0. \tag{2}$$

Definition of Riemann-Liouville's Fractional Integral [10]

Suppose α is a real non- negative number and $f(t)$ is piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$. The Riemann- Liouville's fractional integral of $v(t)$ of order $\alpha > 0$, for $\zeta > 0$ is defined as

$${}_{a}I_t^{\alpha} v(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{\alpha}^{t} \frac{v(\zeta)}{(t-\zeta)^{1-\alpha}} d\zeta, & \alpha > 0 \\ f(t) \qquad\qquad\qquad , & \alpha = 0 \end{cases} \tag{3}$$

Definition of Caputo's Fractional Derivative [10]

If $v \colon \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in (n-1, n), n \in \mathbb{N}$, then

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$$ {}_{0}^{c}D^{\alpha}v(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-\zeta)^{n-1-\alpha}\ \ v^{(n)}\,(\zeta)\,d\zeta \qquad (4) $$

Definition of Laplace Transform for Riemann- Liouville and Caputo fractional **[6]**

The Laplace transform is a powerful tool for solving a class of linear fractional differential equations

1. Laplace transform for Riemann-Liouville's Fractional

$$ \mathcal{L}[{}_0D_t^{\alpha}h(t)] = S^{\alpha}\mathcal{L}[{}_0D_t^{\alpha}h(t)] - \sum_{k=0}^{m-1} S^{k}\ {}_0D_t^{\alpha-k-1}\ h(t)|_{\ t=0}\ , \qquad (5) $$

$$ m-1 \le \alpha < m \in \mathbb{Z}^{+}\ , t>0 $$

2. While Laplace Caputo

$$ \mathcal{L}[{}_0D_t^{\alpha}h(t)] = \ S^{\alpha}\mathcal{L}\ [{}_0D_t^{\alpha}h(t)] - \textstyle\sum_{k=0}^{m-1} S^{\alpha-K-1}\ h^{k}\ (0), \qquad (6) $$

$$ m-1 < \alpha\ \le m \in\ \mathbb{Z}^{+}, t>0 $$

Definition: Mittag-Leffler function **[13]**

The Mittag-Leffler function is a generalisation of the exponential function, and it is one of the most important functions, and represents a key tool in fractional differential equations. The two-parameter Mittag- Leffler function is defined as

$$ E_{r,s}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(rk+s)'} \qquad (7) $$

Where $r, s > 0$. Moreover

1. For $r = s$ in equation (7) simplifies to

$$ E_{r,r}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{(rk+r)'} \qquad r>0 \qquad (8) $$

2. If $s = 1$ in equation (7), then

$$ E_{r,1}(z) = \textstyle\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(rk+1)'} \qquad r>0. \qquad (9) $$

3. If $r = 1$ and $s = 1$ in equation (7), the Mittag-Leffler function reduces to the exponential function

$$ E_{1,1}\,(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^{k}}{k!} = e^{z} \qquad (10) $$

Relations **[6]**

1. For all $r > 0, s > 0$, then $E_{r,s} = zE_{r,r+s}(z) + \frac{1}{\Gamma(r)}$

Proof

By equation (7), put $k = n + 1$,

$$ \begin{aligned} E_{r,s}(z) &= \sum_{n=-1}^{\infty}\frac{z^{n+1}}{\Gamma(r(n+1)+s)} \\ &= \sum_{n=-1}^{\infty}\frac{z^{n+1}}{\Gamma(rn+r+s)} \\ &= \frac{z^{0}}{\Gamma(s)} + \sum_{n=0}^{\infty}\frac{z^{n+1}}{\Gamma(rn+r+s)} \end{aligned} $$

$= \frac{1}{\Gamma(s)} + zE_{r,r+s}(z)$

2. The fractional derivative of $E_{r,s}(z) = \frac{1}{rz}E_{r,s-1}(z) - \frac{s-1}{rz}E_{r,s}(z)$

Properties for some specific values of $r\ and\ s$: the Mittag -Leffler function simplifies to a familiar form. For example [5]

1. $E_{1,2}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k+2)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!} = \frac{e^{z}-1}{z}$
2. $E_{2,1}(z^{2}) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} = \cosh z$
3. $E_{2,2}(z^{2}) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{z(2k+1)!} = \frac{\sinh z}{z}$
4. $E_{2,1}(-z^{2}) = \sum_{k=0}^{\infty}\frac{(-z^{2})^{k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{(-1)^{k}z^{2k}}{(2k)!} = \cos z$
5. $E_{2,2}(-z^{2}) = \sum_{k=0}^{\infty}\frac{(-z^{2})^{k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty}\frac{(-1)^{k}z^{2k+1}}{\Gamma(2k+1)!} = \frac{\sin z}{z}$

Homogeneous and non- homogeneous systems of fractional equations and real symmetric matrices

The principal goal in this section is to apply this method with real symmetric matrices to solve systems of fractional differential equations, as mentioned in the papers [1], [4]. Let the fractional differential system be

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$$D^{\alpha}X(t) = AX(t) + h(t) \tag{11}$$

Such that is a continuous, vector-valued function of the variable t for the unknown vector function $X(t) = \begin{pmatrix} x_{1(t)} \\ \vdots \\ x_{n(t)} \end{pmatrix}$ matrix, as it is a symmetric matrix.

Suppose that $v_1, \ldots, v_n$ are the corresponding eigenvectors orthonormal to the eigenvalues $\lambda_1, \ldots . \lambda_n$

Putting $X(t) = PY(t)$, such that $P = [v_1 \quad \cdots\cdots \quad v_n]$ orthonormal matrix, then

$$D^{\alpha}PY(t) = A\big(PY(t)\big) + h(t) \tag{12}$$

$$PD^{\alpha}Y(t) = APY(t) + h(t) \tag{13}$$

$$D^{\alpha}Y(t) = P^{-1}APY(t) + P^{-1}\,h(t) \tag{14}$$

$$D^{\alpha}Y(t) = DY(t) + P^{T}h(t) \tag{15}$$

Such that D is the diagonalised matrix of A. Also, letting $g(t) = P^{T}h(t)$

$$D^{\alpha}Y(t) = DY(t) + g(t) \tag{16}$$

$$\begin{pmatrix} y_1^{\alpha} \\ \vdots \\ y_n^{\alpha} \end{pmatrix} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots\,\lambda_2 & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix} \tag{17}$$

Then

$$D^{\alpha}Y_i = \lambda_i Y_i + g_i(t) \tag{18}$$

We taking Laplace transform, we get

$$\mathcal{L}\{D^{\alpha}y_i\} = \mathcal{L}\big\{D^{n}\big[D^{-(n-\alpha)}y_i\,\big]\big\} \tag{19}$$

$$= s^{n}\mathcal{L}\big\{D^{-(n-\alpha)}\,y_i\big\} - \sum_{i=1}^{n} s^{n-i}\,D^{i-1-(n-\alpha)}\,y_i(0) \tag{20}$$

$$= s^{n}\big[s^{-(n-\alpha)}Y_i(s)\big] - \sum_{i=1}^{n} s^{n-i}\,D^{i-1-(n-\alpha)}\,y_i(0) \tag{21}$$

$$= s^{\alpha}Y_i(s) - \sum_{i=1}^{n} s^{n-i}D^{i-1-n+\alpha}\,y_i(0) \tag{22}$$

Then, by taking the inverse Laplace transform, we can obtain the solution.

Example 3.2

Solve the non-homogeneous system of fractional-order systems when $\alpha = \frac{1}{5}$

$$X^{\alpha} = \begin{pmatrix} x_1^{\alpha} \\ x_2^{\alpha} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Replacing $X = py$ forms a simple system, with $P = (v_1 \quad v_2)$

The eigenvalues λ of A satisfy when $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 1)$$

Hence, $\lambda_1 = 1, \lambda_2 = 6$

When, $\lambda_1 = 1$

$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\gamma_1 = 1, \gamma_2 = -2$

Thus $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

When, $\lambda_2 = 6$

$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\beta_1 = 2, \beta_2 = 1$

Thus $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Substituting $x = py$ into the differential equation produces

$D^{\alpha}y = p^{-1}Apy + p^{-1}h = Dy + p^{T}h$

$$\begin{pmatrix} y_1^{\frac{1}{5}} \\ y_1^{\frac{1}{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 + \frac{1}{\sqrt{5}}(t-2) \\ 6y_2 + \frac{1}{\sqrt{5}}(2t+1) \end{pmatrix}$$

Now, the first row gives

$D^{\frac{1}{5}}y_1 = \frac{1}{\sqrt{5}}(t-2)$ by equation (5), we get

$s^{\frac{1}{5}}Y_1(s) - D^{-\left(1-\frac{1}{5}\right)}y_1(0) = \dfrac{1}{\sqrt{5}}\left(\dfrac{1}{s^2} - \dfrac{2}{s}\right)$

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$s^{\frac{1}{5}}Y_1(s) - c = \frac{1}{\sqrt{5}}\left(\frac{1}{s^2} - \frac{2}{s}\right)$ by equation (7), we get

$$y_1(t) = c_1 t^{\frac{-4}{5}} E_{\frac{1}{5},\frac{1}{5}}\left(t^{\frac{1}{5}}\right) + \frac{1}{\sqrt{5}} t^{\frac{6}{5}} E_{\frac{1}{5},\frac{11}{5}}\left(t^{\frac{1}{5}}\right) - \frac{2}{\sqrt{5}} t^{\frac{1}{5}} E_{\frac{1}{5},\frac{6}{5}}\left(t^{\frac{1}{5}}\right)$$

Similarly

$$y_2(t) = c_2 t^{\frac{-4}{5}} E_{\frac{1}{5},\frac{1}{5}}\left(6t^{\frac{1}{5}}\right) + \frac{2}{\sqrt{5}} t^{\frac{6}{5}} E_{\frac{1}{5},\frac{11}{5}}\left(6t^{\frac{1}{5}}\right) + \frac{1}{\sqrt{5}}\ t^{\frac{1}{5}} E_{\frac{1}{5},\frac{6}{5}}\left(6t^{\frac{1}{5}}\right)$$

This example investigates the analytical solution of a non-homogeneous system of fractional differential equations of order $\frac{1}{5}$; subsequently, the fractional Laplace transform is applied to solve the resulting equations. Relies on the two- parameter Mittag-Leffler $E_{\alpha,\beta}$, which serves as the crucial mathematical tool for deriving the explicit solutions for the variables $y_1(t)$ and $y_2(t)$

Example 3.3

Solve the homogeneous system of fractional-order system when $\alpha = \frac{2}{3}$

$$X^\alpha = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \\ x_3^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 3 & -3 \\ 1 & -1 & 1 \\ -1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ax$$

Replacing $x = py$ forms a simple system, with $p = (v_1 \quad v_2 \quad v_3)$

The eigenvalues λ satisfy $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 & -3 \\ 1 & -1-\lambda & 1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 2)(\lambda - 4) = 0$$

Hence $\lambda_1 = 1,\ \lambda_2 = -2,\ \lambda_3 = 4$

When $\lambda_1 = 1$

$$\begin{pmatrix} 0 & 3 & -3 \\ 1 & -2 & 1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\ , \gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$$

Thus $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Similarly λ_2, λ_3 we get $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Hence

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

Then

$$p^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Substituting $x = py$

$$D^\alpha y = p^{-1}Apy = Dy$$

$$\begin{pmatrix} y_1^{\frac{2}{3}} \\ y_2^{\frac{2}{3}} \\ y_3^{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

First row yields

$$\begin{pmatrix} y_1^{\frac{2}{3}} \\ y_2^{\frac{2}{3}} \\ y_3^{\frac{2}{3}} \end{pmatrix} = \begin{pmatrix} y_1 \\ -2y_2 \\ 4y_3 \end{pmatrix}$$

By Laplace transform, we get

$$\mathcal{L}\left\{y_1^{\frac{2}{3}}\right\} = \mathcal{L}\{y_1\}$$

$s^{\frac{2}{3}}Y_1(s) - c_1 = \ Y_1(s)$ where $c_1 = D^{-\left(1-\frac{2}{3}\right)}\, y_1(0)$

$$Y_1(s) = \frac{c_1}{s^{\frac{2}{3}} - 1}$$

By equation (7), we get

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$y_1(t) = c_1 t^{-\frac{1}{3}} E_{\frac{2}{3},\frac{2}{3}}\left(t^{\frac{2}{3}}\right)$

Similarly

$y_2(t) = c_2 t^{\frac{-1}{3}} E_{\frac{2}{3},\frac{2}{3}}\left(-2t^{\frac{2}{3}}\right)$

$y_3(t) = c_3 t^{\frac{-1}{3}} E_{\frac{2}{3},\frac{2}{3}}\left(4t^{\frac{2}{3}}\right)$

Conclusion

Currently, it is of paramount importance to study differential equations and fractional operations in the matrix framework. Further, we present analytical techniques for solving fractional differential equation systems with real symmetric matrices using Laplace transform and Mittag-Leffler function methods with some illustrative examples

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