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Original article

Existence and Uniqueness of Solutions for the Bagley–Torvik Problem Involving the Hilfer Fractional Derivative

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Abstract
In this paper, we investigate the existence and uniqueness of solutions for the Bagley–Torvik equation involving the Hilfer fractional derivative. By transforming the original fractional integro-differential equation into an equivalent integral equation, the problem is reformulated in a framework suitable for fixed-point analysis. The existence of at least one solution is established by applying the Schauder fixed-point theorem, while the uniqueness of the solution is obtained under an appropriate Lipschitz-type condition using the Banach contraction principle. The proposed approach provides a rigorous theoretical framework for analyzing Bagley–Torvik equations with Hilfer fractional derivatives.
Keywords. Bagley–Torvik Equation, Hilfer Fractional Derivative, Fractional Calculus, Fixed-Point Theory.

Introduction and Preliminaries

Bagley–Torvik-type problems have been extensively studied. One can find detailed accounts of recent work involving these problems and the following operators in [10,14].
Caputo, Riemann–Liouville, Hadamard, Hadamard–Caputo, and generalized fractional derivatives. We propose considering the Hilfer fractional derivative operator, which encompasses the Caputo and the Riemann–Liouville operators and has received significant attention because it is involved in the equations under investigation.
In this paper, we will talk about a topic that has been recently circulated and has received increasing attention, which is the Bagley-Torvik problem, as it is considered one of the most important fractional differential equations, combining integer derivatives and fractional derivatives. The Bagley-Torvik derivatives at the same time. The problem is also a generalization of both the Caputo derivative and the Riemann-Liouville derivative.

Definition 1

Bagley-Torvik problem is a second-order differential equation that contains a fractional derivative, and its known formula:

$$aD^2x(t) + bD^{3/2}x(t) + cx(t) = f\big(t, x(t)\big)$$

Initial conditions

$$x(0) = x_0\,, \qquad x'(0) = x_1$$

where $D^{3/2}$ is a fractional derivative. It was chosen $\alpha = 3/2$ not random. because this test provides a suitable mathematical framework for analyzing the equation and allows converting it into an integral equation and studying the existence of a solution for it using the fixed point theory.

Definition 2

Let $\alpha \in (1,2), \beta \in [0,1], f \in L_1\ and\ \big(I^{(1-\beta)(2-\alpha)}f\big)(t) \in AC^2(I)$ The Hilfer fractional derivative of order $\alpha\ and$ type β of f is defined as:

$$\big(D^{\alpha,\beta}f\big)(t) = \left(I^{\beta(2-\alpha)}\frac{d^2}{dt^2}I^{(1-\beta)(2-\alpha)}\,f\right)(t)\ for\ a.e.\ \ t \in I, \quad 1 < \alpha < 2.$$

There are special cases that must be mentioned
When $\beta = 0$ we obtain the Riemann-Liouville derivative:

$$D^{\alpha,0}f(t) = \frac{d^2}{dt^2}I^{(2-\alpha)}f(t).$$

When $\beta = 1$ we obtain the Caputo derivative:

$$D^{\alpha,1}f(t) = I^{(2-\alpha)}\frac{d^2}{dt^2}f(t).$$

Therefore, the initial conditions with Hilfer are given in the following form:

$$I^{(1-\beta)(2-\alpha)}x(0) = x_0\,, \quad \frac{d}{dt}\big(\,I^{(1-\beta)(2-\alpha)}x\big)(0) = x_1.$$

Theorem 1.1. [8]

The fractional integral operators send $\mathrm{L_q}[0, \mathrm{b}]$ continuously into $\mathrm{L_p}[0, \mathrm{b}]$ if $\mathrm{p} \in [1, \infty]$satisfy $\mathrm{q} > 1/(\alpha + (1/\mathrm{p}))$ (a deep result from interpolation theory implies that even $\mathrm{q} = 1/(\alpha + (1/\mathrm{p}))$ is allowed if $1 < \mathrm{p} < \infty$). In particular, $\mathrm{I}^{\alpha}\colon \mathrm{L_p}[0, \mathrm{b}] \to \mathrm{L_p}[0, \mathrm{b}]$ is compact for each $\mathrm{p} \in [1, \infty]$.

***Theorem 1.2.* (Kolmogorov Compactness Criterion) [6].**

The bounded subset Ω of $L_p(0,1)$ is relatively compact if, and only if

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$$\lim_{h\to 0}\|T_h x - x\|_{L_p} = 0,$$

uniformly on Ω, that is

$$\lim_{h\to 0}\left\{\begin{matrix} sup \\ x\in\Omega\end{matrix}\|T_h x - x\|_{L_P}\right\} = 0, \text{where } T_h x(t) = \frac{1}{h}\int_t^{t+h} x(s)\,ds.$$

***Theorem 1.3.* (Schauder fixed point theorem) [4]**

Let U be a convex subset of a Banach space F, and $T: U \to U$ is a compact, continuous map (that is, a completely continuous map). Then T has at least one fixed point in U.

In other words:

Let U be a compact convex subset of a Banach space F, and $T: U \to U$ is a continuous map. Then T has at least one fixed point in U.

Theorem 1.4. (Banach fixed point theorem) [4, 6]

Let $(U, \|.\|)$ be a nonempty complete normed space. Let $T: U \to U$ be a map such that for every $u, v \in U$ the relation

$$\|T_u - T_v\| \le \lambda\|u - v\|, \quad 0 \le \lambda < 1\,,$$

holds. Then the operator Thas a unique fixed point $u \in U$.

Lemma 1.

Let $\beta \in (m-1, m), m \in \mathbb{N}$ and $\alpha \ge \beta$. If f is a function such that $I^{m-\beta} f \in AC^m[0, b]$, then

$$I^\alpha D^\beta f = \mathrm{I}^{\alpha-\beta} f - \sum_{k=0}^{m-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)}\left[D^{m-k-1} I^{m-\beta} f(t)\right]_{t=0} \tag{1.1}$$

In particular

$$I^\alpha D^\alpha f = f - \sum_{k=0}^{m-1} \frac{t^{\alpha-k-1}}{\Gamma(\alpha-k)}\left[D^{m-k-1} I^{m-\alpha} f(t)\right]_{t=0} \tag{1.2}$$

If $\alpha \in (0,1)$ it follows

$$I^\alpha D^\alpha f = f - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} f(0) = f, \qquad if\ f \in C[0, b].$$

We conclude this section by stating the following fixed point theorem, which is a trivial application of the fixed point index of a compact operator $T: Q \to K$ in the "Absolute neighborhood retract" of a closed convex cone K on Q where $Q = B_{r_0} \cap K$ (for details, cf. e.g. ([4, 23, 24]).

Note that converting the Bagley-Torvik equation from the fractional differential form to the integral form allows the use of functional analysis tools such as fixed point theory to prove the existence of a solution.

Existence and Uniqueness for Bagley-Torvik problem with Hilfer operator

We address the existence of integrable solutions for the fractional Bagley-Torvik problem represented by a polynomial fractional differential equation formulated as follows

$$\begin{cases} AD^{2,\beta} x(t) + BD^{3/2,\beta} x(t) + Cx(t) = f(t, x(\varphi(t))) \\ I^{(1-\beta)(2-\alpha)} x(0) = x_0 \\ \dfrac{d}{dt} I^{(1-\beta)(2-\alpha)} x(0) = x_1 \end{cases} \tag{2.1}$$

Where $t \in (0,1), \alpha = 3/2\,, 0 \le \beta \le 1,\ n-1 < \alpha < n$.

We now convert Equation (1) into an integral equation. We move $BD^{3/2,\beta}x(t) + Cx(t)$ to the other side, so it becomes:

$$AD^{2,\beta} x(t) = -BD^{3/2,\beta} x(t) - Cx(t) + f(t, x(\varphi(t)))$$

We divide both sides of the equation by A

$$D^{2,\beta} x(t) = \frac{1}{A} f\left(t, x(\varphi(t))\right) - \frac{B}{A} D^{3/2,\beta} x(t) - \frac{C}{A} x(t)$$

We now apply the fractional integration operator I^2 to both sides

$$I^2 D^{2,\beta} x(t) = \frac{1}{A} I^2 f\left(t, x(\varphi(t))\right) - \frac{B}{A} I^2 D^{3/2,\beta} x(t) - \frac{C}{A} I^2 x(t)$$

We now calculate $I^2 D^{2,\beta} x(t)$ using the Hilfer property

$$I^\alpha D^{\alpha,\beta} x(t) = x - \sum_{k=0}^{n-1} \frac{t^k}{k!}\left[\frac{d^k}{dt^k} I^{(1-\beta)(n-\alpha)} x\right]_{t=0}$$

Since $n = 2\,, \alpha = 2, for$

$$(1-\beta)(n-\alpha) = (1-\beta)(2-2) = 0$$

Then

$$I^{(1-\beta)(2-2)} = I^0 = I$$

So, it becomes

$$I^2 D^{2,\beta} x(t) = x(t) - x(0) - tx'(0)$$

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And among the initial conditions $x(0) = x_0, x'(0) = x_1$.

$$I^2D^{2,\beta}x(t) = x(t) - x_0 - tx_1$$

$I^2D^{3/2,\beta}x(t)$ account, we have $\alpha = 3/2, n = 2$

Using the property of composition of integrals and derivatives, we obtain:

$$I^2D^{3/2} = I^{1/2} \ any\ I^2D^{3/2,\beta}x(t) = I^{1/2}x(t)$$

We group the terms.

$$x(t) - x_0 - tx_1 = -\frac{B}{A}I^{1/2}x(t) - \frac{C}{A}I^2x(t) + \frac{1}{A}I^2f(t,x(\varphi(t)))$$

$$x(t) = x_0 + tx_1 = -\frac{B}{A}I^{1/2}x(t) - \frac{C}{A}I^2x(t) + \frac{1}{A}I^2f(t,x(\varphi(t)))$$

Our investigation is based on reducing the problem formally (1) into the Volterra integral equation (cf. Lemma 1)

$$x(t) = x_0 + tx_1 - \frac{B}{A\boldsymbol{\Gamma}(1/2)}\int_0^t (t-s)^{-1/2}x(s)ds - \frac{C}{A\boldsymbol{\Gamma}(2)}\int_0^t (t-s)x(s)ds + \frac{1}{A\boldsymbol{\Gamma}(2)}\int_0^t (t-s)f\left(s,x(\varphi(s))\right)ds.$$

$$x(t) = x_0 + tx_1 - \frac{B}{A\boldsymbol{\Gamma}(1/2)}\int_0^t (t-s)^{-1/2}x(s)ds$$
$$-\frac{C}{A}\int_0^t (t-s)x(s)ds + \frac{1}{A}\int_0^t (t-s)f\left(s,x(\varphi(s))\right)ds. \qquad (2.2)$$

This is the Volterra integral equation.

Assumptions

1. Let $f:(0,1)\times\mathbb{R}\to\mathbb{R}$ satisfy the Carathéodory conditions.
(a) For every $t\in(0,1)$, $f(t,.)$ is continuous.
(b) For every $x\in\mathbb{R}$, $f(.,x)$ is measurable.
(c) There is a function $a\in L_1(0,1)$ and a constant $b>0$ such that $|f(t,x)| \le a(t) + b(t)|x|$, for each $t\in(0,1), x\in\mathbb{R}$,
2. Let $\varphi:(0,1)\to(0,1)$ is nondecreasing, absolutely continuous, and there is a constant $\mathrm{M}>0$ such that $\varphi'(t) \ge M\ a.e\ on\ (0,1)$.
3. $A \ne 0$, $There\ is\ r > 0$ such that $M + kr \le r$.

Where $M = |x_0| + \frac{|x_1|}{2} + \frac{\|a\|}{2|A|}$ *and* $k = \frac{2|B|}{|A|\Gamma(1/2)} + \frac{|C|}{2|A|} + \frac{b}{2|A|M}$.

Thus, we are in a position to formulate and prove the following.

Theorem 2.1.

Assume that hypotheses 1–2 are satisfied. Then the fractional Bagley–Torvik problem (2.1) admits at least one solution $x \in L_1(0,1)$. Furthermore, if

$$\frac{2|B|}{|A|\Gamma(1/2)} + \frac{|C|}{2|A|} + \frac{sup|b(t)|}{2|A|M} < 1, \qquad (2.3)$$

then Problem (2.1) possesses a unique solution in $L_1(0,1)$. Moreover, by the integral representation property of the Hilfer fractional derivative, Problem (2.1) is equivalent to the integral equation (2.2). Consequently, we define the operator

$$T: L_1(0,1) \to L_1(0,1)$$

by

$$(Tx)(t) = x_0 + tx_1$$
$$-\frac{B}{A\boldsymbol{\Gamma}(1/2)}\int_0^t (t-s)^{-1/2}x(s)ds$$
$$-\frac{C}{A}\int_0^t (t-s)x(s)ds + \frac{1}{A}\int_0^t (t-s)f\left(s,x(\varphi(s))\right)ds. \qquad (2.4)$$

We demonstrate that

$$T: L_1(0,1) \to L_1(0,1)$$

Based on the hypothesis $|f(t,x)| \le a(t) + b(t)|x|$

And subsequently

$$\|Tx\| \le |x_0| + \int_0^1 |x_1 t|\,dt + \frac{1}{|A|}\int_0^1 \left|\int_0^t (t-s)\left|f\left(s,x(\varphi(s))\right)\right|ds\right|dt$$
$$+\frac{|B|}{|A|\Gamma(1/2)}\int_0^1 \left|\int_0^t (t-s)^{-1/2}x(s)ds\right|\,dt - \left|\frac{C}{A}\right|\int_0^1 \left|\int_0^t (t-s)|x(s)|\,ds\right|dt$$
$$\le |x_0| + \frac{|x_1|}{2} + \int_0^1\int_0^t (t-s)\left|f\left(s,x(\varphi(s))\right)\right|\,\mathrm{dsdt}$$
$$+\frac{|B|}{|A|\Gamma(1/2)}\int_0^1\int_0^t (t-s)^{-1/2}|x(\varphi(s))|ds\ dt - \left|\frac{C}{A}\right|\int_0^1\int_0^t (t-s)\left|x(\varphi(s))\right|ds\,dt$$

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$$\leq |x_0| + \frac{|x_1|}{2} + \int_0^1 \int_0^t (t-s)\left|a(t) + b(t)|x(\varphi(s))|\right| \text{ dsdt}$$
$$+ \frac{|B|}{|A|\Gamma(^1/_2)} \int_0^1 \int_0^t (t-s)^{-1/2} \left|x(\varphi(s))\right| ds \ dt - \left|\frac{C}{A}\right| \int_0^1 \int_0^t (t-s)\left|x(\varphi(s))\right| ds\, dt$$
$$\leq |x_0| + \frac{|x_1|}{2} + \int_0^1 \int_0^t (t-s)a(t) \text{ dsdt} + \int_0^1 \int_0^t (t-s)\, b(t)|x(\varphi(s))| \text{ dsdt}$$
$$+ \frac{|B|}{|A|\Gamma(^1/_2)} \int_0^1 \int_0^t (t-s)^{-1/2} \left|x(\varphi(s))\right| ds \ dt - \left|\frac{C}{A}\right| \int_0^1 \int_0^t (t-s)\left|x(\varphi(s))\right| ds\, dt$$

Interchanging the order of integration implies

$$\int_0^1 \int_0^t (t-s)^{-1/2} \left|x(\varphi(s))\right| ds \ dt = \int_0^1 |x(\varphi(s))| (\int_s^1 (t-s)^{1/2}) dtds$$
$$= \int_0^1 |x(\varphi(s))| \left. \frac{(t-s)^{1/2}}{^1/_2} \right|_s^1 ds = 2(1-s)^{1/2} \leq 2.$$

Analogously, we have

$$\int_0^1 \int_0^t (t-s)\left|x(\varphi(s))\right| ds \ dt = \int_0^1 |x(\varphi(s))| (\int_s (t-s)) dtds$$
$$= \int_0^1 |x(\varphi(s))| \frac{(1-s)^2}{2} ds \leq \frac{1}{2} \, \|x\|$$

Also, from the hypothesis $\varphi'(t) \geq M > 0$

$$\int_0^1 \int_0^t (t-s)a(t) \text{ dsdt} + \int_0^1 \int_0^t (t-s)\, b(t)|x(\varphi(s))| \text{ dsdt} = \frac{\|a\|}{2}$$
$$+ \frac{1}{2} sup|b| \int_0^1 |x(\varphi(s))| ds \ = \frac{\|a\|}{2} + \frac{sup|b|}{2M} \, \|x\|$$

Consequently

$$\|Tx\| \leq |x_0| + \frac{|x_1|}{2} + \frac{1}{2|A|}\|a\| + \frac{sup|b(t)|}{2|A|M}\|x\| + \frac{2|B|}{|A|\Gamma(1/2)}\|x\| + \frac{|C|}{2|A|}\|x\|$$

All terms on the right side are finite because $x \in L_1(0,1)$, $a \in L_1(0,1)$. Than $\|Tx\| \leq \infty$
And consequently, $Tx \in L_1(0,1)$ and

$$T: L_1(0,1) \to L_1(0,1)$$

Let's take $B_r = \{x \in L_1(0,1): \|x\| \leq r\}$
At that point $\|Tx\| \leq C_0 + K$ where:

$$C_0 = |x_0| + \frac{|x_1|}{2} + \frac{\|a\|}{2|A|} \ \ and \ K = \frac{2|B|}{|A|\Gamma(1/2)} + \frac{|C|}{2|A|} + \frac{sup|b(t)|}{2|A|M}$$

From (2.3) choose $r > 0$ such that $C_0 + K \leq r$. Hence, $T(B_r) \subseteq B_r$.
If $x_n \to x \ in \ L_1(0,1)$, then from the Carathéodory conditions, we obtain $Tx_n \to Tx$ Therefore, T is continuous. Since the integral kernel $(t-s), (t-s)^{-1/2}$ is integrable, $T(B_r)$ is a bounded and equi-integrable set. Thus, using Kolmogorov (1.2) in L_1.
$T(B_r)$ is a relatively compact set. if T is a completely compact operator. Since B_r is a convex and closed set, $T(B_r) \subseteq B_r, T$ is a continuous and completely compact operator, then Schauder's theorem (1.3) guarantees the existence of an $x \in B_r$ such that $Tx = x$ meaning x is a solution to the integral equation and, consequently, a solution to the Bagley-Torvik problem involving the Hilfer derivative.
Now, to prove the uniqueness of the solution, we add to the hypotheses of the antecedent of the Lipschitz condition that there is a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|, \qquad t \in (0,1), \qquad x, y \in \mathbb{R}.$$

And that:

$$\frac{2|B|}{|A|\Gamma(1/2)} + \frac{|C|}{2|A|} + \frac{L}{2|A|M} < 1.$$

Proof: Let $x, y \in L_1(0,1)$ and we have
We take the standard

$$\|Tx - Ty\| = \int_0^1 |(Tx)(t) - (Ty)(t)| dt$$

And using the triangle inequality

$$\|Tx - Ty\| \leq \frac{|B|}{|A|\Gamma(1/2)} \int_0^1 |\, x(s) - y(s)| \left(\int_s^1 (t-s)^{-1/2} dt \right) ds$$
$$+ \frac{|C|}{|A|} \int_0^1 \int_0^t (t-s)|x(s) - y(s)| dsdt$$

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$$+\frac{L}{|A|}\int_0^1\int_0^t (t-s)|x(\varphi(s))-y(\varphi(s))|\,dsdt$$
$$\leq \frac{2|B|}{|A|\Gamma(1/2)}\|x-y\|+\frac{|C|}{2|A|}\|x-y\|+\frac{L}{2|A|}\int_0^1|x(\varphi(s))-y(\varphi(s))|ds$$

From hypothesis $\varphi\,'(t)\geq M>0$ we obtain:

$$\|Tx-Ty\|\leq \frac{2|B|}{|A|\Gamma(1/2)}\|x-y\|+\frac{|C|}{2|A|}\|x-y\|+\frac{L}{2|A|M}\|x-y\|$$
$$\leq\left[\frac{2|B|}{|A|\Gamma(1/2)}+\frac{|C|}{2|A|}+\frac{L}{2|A|M}\right]\|x-y\|$$

From the Lipschitz condition

$$\|Tx-Ty\|\leq k\|x-y\|$$

We have

$$k=\frac{2|B|}{|A|\Gamma(1/2)}+\frac{|C|}{2|A|}+\frac{L}{2|A|M}\,.$$

And from the hypothesis $k<1$, if T is a contraction mapping on $L_1(0,1)$, then by the Banach fixed point Theorem (1.4), there exists a unique element $x\in L_1(0,1)$ such that $Tx=x$, That is, the integral equation has a unique solution, and consequently, (2.1) has a unique solution. which completes the proof.

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