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Original article

On Generalized δ- Pre-Closed Set in Topological Spaces

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Abstract
In his 1970 [1] work, Levine introduced generalized closed sets (g-closed), from which the notion of generalized open sets (g-open) follows by defining them as the complements of generalized closed sets. This study presents and examines a newly defined class of sets, denoted by generalized δPre–closed sets (gδP- closed). It builds upon our earlier research on generalized semi α- closed sets (gs α- closed) [2] by extending the underlying framework and introducing a different generalized class. The paper establishes several essential properties of the proposed sets, including their behavior with respect to intersections, unions, and set inclusion, and investigates their connections with related generalized topological concepts. To demonstrate the validity of the theoretical results, a collection of illustrative examples together with counterexamples is provided.
Keywords: Closed Set, G- Closed, P- Closed, δ- Closed, δg- Closed, gδ- Closed, gδP- Closed.

Introduction

Closed sets and their generalized variants constitute a fundamental cornerstone in the study of topological spaces. It is noteworthy that a topology on a given set can be fully characterized by prescribing the behavioral properties of both its closed sets and generalized closed sets. The idea of reinforcing the notions of open and closed sets can be traced back to the early work of Stone [3], which gave rise to what are now known as regular open and regular closed sets. Levine later expanded on this direction by introducing a new class of sets termed generalized closed sets [1], an initiative that proved highly influential in stimulating various research lines concerned with exploring weaker forms of closed sets within general topology. This line of inquiry has led to a considerable accumulation of scholarly work, with multiple researchers directing their efforts toward studying the behavior and properties of these diverse variants across different topological settings, thereby enriching the theory and broadening its scope. Several researchers have contributed to the development of generalized closed sets. For instance, Dontchev et al. [4], Sudha et al. [5], and Meena et al. [6] each introduced different forms of g-closed sets by making use of δ- closed sets.

The primary objective of this paper is to further develop the theory of generalized δPre –closed sets (gδP- closed) by introducing a new perspective and systematically investigating their topological properties. We first establish the definition of generalized δPre–closed sets (gδP- closed) and then examine a collection of their basic and structural properties within topological spaces (X,τ). In addition, the relationships between generalized δPre – closed sets (gδP-closed) and other generalized closed sets are explored. The results presented in this study contribute to a deeper understanding of these generalized structures and demonstrate their importance in the ongoing development of general topology.

Preliminaries

Throughout this work, we denote a topological space by X or (X,τ). Unless otherwise specified, no separation axioms are assumed. The following definitions will be used in subsequent sections.

***Definition* [7]**
Let (X,τ) be a topological space, and let A is a non-empty subset of X :

- The closure of A, denoted $cl(A)$, is the intersection of all closed sets containing A.
- The interior of A denoted $int(A)$, is the union of all open sets contained in A.

***Definition* [8]**
A subset A of (X,τ) is called a Regular open set if $A = \mathrm{int}(\mathrm{cl}(\mathrm{A}))$.

***Definition* [1]**
A subset A of a topological space (X,τ) is called generalized closed (g- closed) if $\mathrm{cl}(\mathrm{A}) \subseteq \mathrm{U}$ whenever $A \subseteq U$ and U is open in (X,τ).

***Definition* [9]**
A subset A of a topological space (X,τ) is called g*- closed set if $\mathrm{cl}(\mathrm{A}) \subseteq \mathrm{U}$ whenever $A \subseteq U$ and U is g-opened in (X,τ).

***Definition* [10]**
Suppose that (X,τ) is a topological space. Then:

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- A subset $A \subseteq X$ is called a pre-closed set (briefly P-closed) if $l\big(int(A)\big) \subseteq A$.
- A subset $A \subseteq X$ is called pre-open set (briefly P-open) if $A \subseteq int\big(cl(A)\big)$.

Definition **[11- 12]**

Let A be a subset of a space X, then:

- The pre-closure sets of A, denoted by $\big(Pcl(A)\big)$, is the intersection of all pre-closed sets (P-closed) containing A.
- The pre-interior of A, denoted by (P-$int(A)$), is the union of all pre-open sets (P-open) contained in A.

Definition **[13]**

Suppose that (X,τ) is a topological space. A subset $A \subseteq X$is said to be a δ- closed set if $A = \delta\mathrm{cl}(A)$, where $\delta\mathrm{cl}(A) = \{x \in X: \mathrm{int}(\mathrm{cl}(U)) \cap A \neq \emptyset, U \in \tau \; and \; x \in U\}$.

Definition **[14]**

Suppose that (X,τ) is a topological space. A subset $A \subseteq X$ is said to be a δg^*- closed if $\delta\mathrm{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ- open in X.

Definition **[14- 15]**

Suppose that (X,τ) is a topological space. A subset $A \subseteq X$is said to be:

- A generalized pre-closed set (briefly –gp closed) if $Pcl(A) \subseteq U$ whenever $A \subseteq U$and$U$ is an open set in X.
- A pre-generalized closed set (briefly Pg-closed) if $Pcl(A) \subseteq U$ whenever$A \subseteq U$ and U is a pre-open set (P-open) in X.

Definition **[4- 16]**

Suppose that (X,τ) is a topological space. A subset $A \subseteq X$is said to be:

- A generalized δ-closed set (briefly gδ- closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$and$U$ is δ- open set in X.
- A δ- generalized closed set (briefly δg-closed) if $cl\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is an open set in X.

Generalized δ-Pre-closed set in topological spaces

Definition

Let (X,τ) be a topological space. A subset $A \subseteq X$ is called:

- Generalized δ- pre-closed sets (briefly. gδp- closed) if $\delta\mathrm{p}\big(\mathrm{cl}(\mathrm{A})\big) \subseteq U$ whenever $A \subseteq U$ and U is an open set in (X,τ).
- Generalized δ- pre-open sets (briefly. gδp- open) if its complement is generalized δ- pre-closed sets (briefly. gδp- closed).
- The collection of all generalized δ- pre-closed sets (briefly. gδp- closed) in (X,τ) is denoted by $\big(\mathrm{G\delta pC(X)}\big)$.

Example

Let $X = \{a,b,c\}$, and $\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$. Then a set $A = \{a,c\}$ is generalized δ- pre-closed sets (briefly. gδp- closed).

Remark In general, a generalized δ- pre-closed sets (briefly. gδp- closed) set in X is not necessarily a closed set in X, as illustrated in the following example.

Example

In Example 3.2. The subset $\{a,b\}$, is generalized δ- preclosed sets (briefly. gδp- closed) set in X, but it does not qualify as a closed set in X.

Remark

From definition 3.1 we observe the following inclusion relation:

- Every pre-closed set in X is generalized δ- pre-closed sets (briefly. gδp- closed) set
- The converse does not hold in general.

Example

In Example 3.2, the subset $\{b\}$, is generalized δ- pre-closed sets (briefly. gδp- closed) set in X but not a pre-closed set in X.

Remark

Definition 3.1 establishes the following relationship

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- Every δ- closed set in X is generalized δ- pre-closed sets (briefly. gδp- closed) set
- The converse is not necessarily true.

Example
In Example 3.2. The subset $\{b, c\}$, is generalized δ- pre-closed sets (briefly. gδp- closed) set in X but not δ- closed set in X.

Remark
From definition 3.1, we observe the following inclusion relation:

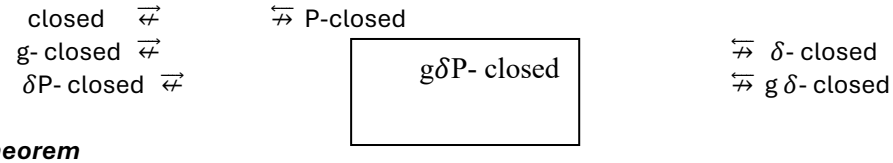
- Every δ- pre-closed set (briefly δp- closed) in X is generalized δ- pre-closed sets (briefly gδp- closed) set in X.
- The converse does not hold in general.

Example
In Example 3.2. The subset $\{c\}$, is generalized δ- pre-closed sets (briefly. gδp- closed) set in X but not δ- pre-closed set (briefly δp- closed) in X.

Remark
In general, a generalized δ- pre-closed sets (briefly gδp- closed) set in X is not necessarily a generalized δ- closed set (briefly gδ- closed) in X, as illustrated in the following example.

Example
In Example 3.2. The subset $\{a, b\}$, is generalized δ- preclosed sets (briefly. gδp- closed) set in X, but it does not qualify as generalized δ- closed set (briefly gδ- closed) closed set in X.

Notes 3.13
Based on Definition 3.1 and the preceding results, the following implication diagram summarizes the relationships among the considered sets.



Theorem
Let (X, τ), be a topological space and $A \subseteq X$. Then the following are equivalent:
a)- Every pre-closed set (briefly. p- closed) of X is generalized δ- pre-closed sets (briefly. gδp- closed).
b)- Every subset of X is generalized δ- pre-closed sets (briefly. gδp- closed).

Proof
(a)→ (b) Let $A \subseteq U$and A is an arbitrary subset of X, where U is open. By (a), then A is generalized δ- pre-closed sets (briefly. gδp- closed) and thus $\delta Pcl(A) \subseteq \delta Pcl(U) = U$. Hence A is generalized δ- pre-closed sets (briefly. gδp- closed) in X.
(b)→ (a) If $U \subset X$ is open, then by (b) $\delta Pcl(A) \subseteq \delta Pcl(U) = U$ and $cl(int(A)) \subseteq A$ for any $A \subseteq X$ or equivalently A is pre closed set (briefly. p- closed) in X.

Remark
(a)- The class of generalized δ- pre-closed sets (briefly. gδp- closed) in X is closed under finite unions.
(b)- The class of generalized δ- pre-closed sets (briefly. gδp- closed) in X is closed under finite intersections.

Example
Based on the results established in Example 3.2, the following example demonstrates the validity of Remark 3.15.

Theorem
The union of two generalized δ- pre-closed sets (briefly. gδp- closed) are generalized δ- pre-closed sets (briefly. gδp- closed).

Proof
Suppose U be an open set in X and $A, B \in (G\delta PC(X))$. Let U containing $A \cup B$, such that $A \subseteq U$ and $B \subseteq U$. Since A and B are generalized δ- preclosed sets (briefly. gδp- closed), then $\delta Pcl(A) \subseteq U$ and $\delta Pcl(B) \subseteq U$. But properties of δP-closure, $\delta Pcl(A \cup B) = \delta Pcl(A) \cup \delta Pcl(B) \subseteq U$. Therefore, $A \cup B$ is generalized δ- pre-closed sets (briefly. gδp- closed).

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Theorem

If A is generalized δ- pre-closed sets (briefly. gδp- closed) in X and $A \subseteq B \subseteq \delta Pcl(A)$, then B is also generalized δ- pr closed sets (briefly. gδp- closed).

Proof

Suppose U be an open set in X, such that $B \subseteq U$, since $A \subseteq B$ we have $A \subseteq U$. Because A is generalized δ- pre-closed sets (briefly. gδp- closed), then $\delta Pcl(A) \subseteq U$. By hypothesis, $B \subseteq \delta Pcl(A)$, so $\delta Pcl(B) \subseteq \delta Pcl(A)$. Therefore, $\delta Pcl(B) \subseteq U$. Thus the set B is generalized δ- pre-closed sets (briefly. gδp- closed) in X.

Theorem

Let $A \subseteq Y \subseteq X$ where A is generalized δ- pre-closed sets (briefly. gδp- closed) in X. Then A is generalized δ- pre-closed sets (briefly. gδp- closed) relative to Y.

Proof

Consider $A \subseteq U \cap Y$, where U is open set in X. Since A is generalized δ- pre-closed sets (briefly. gδp- closed) in X, $A \subseteq U$ which implies $\delta Pcl(A) \subseteq U$. That is $Y \cap \delta Pcl(A) \subseteq U \cap Y$. Where $Y \cap \delta Pcl(A)$ is the δP- closure of A. Thus A is generalized δ- pre-closed sets (briefly. gδp- closed) relative to Y.

Conclusion

This paper examines the concept of(gδ-P closed) sets in topological spaces by investigating their basic properties and discussing their main characteristics. Several examples are included to illustrate the concept and to provide a better understanding of its behavior in different topological settings.

The results show that (gδ-P closed) sets possess several useful properties that can be employed to establish new results related to connectedness, compactness, and other topological concepts. These properties also demonstrate the importance of (gδ-P closed) sets as a useful tool for further studies in topology.

In addition, the study indicates that the concept of (gδ-P closed) sets can be considered in different classes of topological spaces, bitopological spaces, and multi topological spaces. This highlights the flexibility of the concept and its potential to support future developments in topological research.

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