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Original article

Numerical Treatment of Sturm-Liouville Problems with Two Interior Transition Points

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Abstract
Boundary value problems (BVPs) for second-order differential equations, particularly Sturm-Liouville equations, are fundamental in modeling numerous phenomena across the natural sciences. A significant challenge arises when these problems involve additional transmission conditions at internal points. This paper presents a numerical framework for solving Sturm-Liouville boundary value problems characterized by two internal transmission points. We extend the Finite Difference Method (FDM) to effectively address such problems, with a particular emphasis on ensuring accuracy at the critical transmission points $x = \pm 1/3$. The methodology, which generalizes the approach for a single discontinuity, is implemented and tested on a mathematical model. Numerical results demonstrate that the proposed scheme successfully handles the derivative discontinuities, and maintains a second-order convergence rate $O(h^2)$, and shows excellent agreement with the exact solution, proving the robustness and reliability of the developed framework for multi-point interaction models.
Keywords. Finite Difference Method, Sturm-Liouville Problem, Boundary Value Problem, Transmission Conditions.

Introduction
Boundary Value Problems (BVPs) for second-order differential equations serve as fundamental tools in the mathematical modeling of various phenomena across the natural sciences and engineering. Among these, Sturm-Liouville equations hold a prominent position in both theoretical and applied mathematics. Their historical origins trace back to the application of Fourier series in heat transfer problems, and they appear prominently in the study of domains involving interfaces or composite materials.
A significant mathematical challenge arises when these problems incorporate additional transmission conditions at specific internal points. Consequently, recent years (2021–2026) have witnessed an increasing interest in the numerical and analytical treatment of Sturm-Liouville problems with discontinuities. Foundational work in this field has been established through the contributions of Mukhtarov et al., who developed modified Finite Difference Methods (FDM) to handle points of discontinuity. Furthermore, Pandey (2021, 2023) explored high-order finite difference methods for boundary value problems involving internal conditions.
This research aims to further develop and generalize this approach by extending the Finite Difference Method to effectively address Sturm-Liouville problems. The current work seeks to broaden the mathematical framework to handle symmetric internal transmission points, specifically at $x = 1/3$, thereby contributing to more accurate solutions for problems involving complex internal interactions.

Methodology
The Finite Difference Method for Sturm-Liouville Problems. The finite difference method is based on converting a differential equation into a system of algebraic equations by replacing derivative expressions with appropriate finite differences. Let us consider the boundary-value problem for the Sturm-Liouville equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \qquad (1)$$

subject to nonhomogeneous boundary conditions:

$$y(a) = \alpha, \qquad y(b) = \beta \qquad (2)$$

The domain $[a, b]$ is discretized into N equal subintervals, such that $x_i = a + ih$ for $i = 0, 1, \dots, N$. The derivatives at an interior node x_i are approximated using the second-order central difference formulas:

y'x≈y(x + h) − y(x − h) 2h 3

and

$$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} \qquad (4)$$

Substituting these into Eq (1) yields a system of linear algebraic equations $M = B$, where M is a tridiagonal matrix. For the interior nodes x_i(excluding the transition points), The discretization leads to a tridiagonal system where the i-th row of the matrix M is constructed from the coefficients of y_{i-1}, y_i, and y_{i+1}
As derived from the central difference operators. This ensures a consistent second-order truncation error $O(h^2)$ throughout the continuous sub-intervals.

Extension to Two Interior Transition Points
This section represents the core scientific contribution by extending the classic FDM to accommodate two interior transition points. We consider the Sturm–Liouville problem:

$$y''(x) + y'(x)\tan x + y(x)\cos^2 x = 0, \qquad x \in [-1,1]$$

with boundary conditions:

$$y(-1) = 0, \qquad y(1) = 1$$

and the transmission conditions at $x = \pm 1/3$

$$y(-1/3+) = y(-1/3-), \qquad y'(-1/3+) = 3y'(-1/3-)$$

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$$y(1/3+) = y(1/3-), \qquad y'(1/3+) = 3y'(1/3-)$$

To verify the accuracy and the convergence of the proposed finite difference scheme, the numerical results are compared against the analytical exact solution. For this specific model, the exact solution $y(x)$ is constructed as a piecewise function that satisfies the jump conditions at the transition points, and is given by:

$$y(x) = c_1\cos(\sin x) + c_2\sin(\sin x), \quad x \in [-1,-1/3)$$
$$c_3\cos(\sin x) + c_4\sin(\sin x), \quad x \in (-1/3,1/3)$$
$$c_5\cos(\sin x) + c_6\sin(\sin x), \quad x \in (1/3,1]$$

To maintain second-order accuracy at the transition points $\xi = \pm 1/3$, We employ one-sided finite difference approximations for the derivative jump conditions. For the condition$y'(\xi^+) = 3y'(\xi^-)$, We use a second-order forward difference formula for the right-hand limit and a second-order backward difference formula for the left-hand limit. This approach prevents the numerical scheme from crossing the discontinuity interface, thereby ensuring that the global $O(h^2)$ Convergence is preserved without stability loss. By applying the central finite difference method (FDM) at a typical grid point x_i, we obtain:

$$(2 - h\tan(x_i)y_{i-1} + (-4 + 2h^2\cos^2(x_i)\,y_i + (2 + h\tan(x_i)y_{i+1} = 0$$

To ensure the accuracy of the solution at the transition points $x = \pm 1/3$, The transmission conditions are directly integrated into the matrix M. At these specific points, the standard central difference formula is replaced by a discrete representation of the transmission condition. Using forward and backward difference approximations, the following linear relationship is derived:

$$y_{k-1} - 4y_k + 3y_{k+1} = 0$$

Where y_k is the solution at the transition point. This formulation allows the system to correctly handle the discontinuity in the first derivative.

Results and Discussion

The proposed finite difference method was implemented in MATLAB. The computations were carried out on a uniform grid over the interval$[-1,1]$ with $N = 500$ intervals, with the two interior transition points located at $x = -1/3$ and $x = 1/3$. (Table1) Presents a comparison between the approximate and exact solutions at selected points across the domain. The results show a high degree of accuracy, with the absolute error remaining consistently small.

Table 1. Comparison of Approximate and Exact Solutions $(N = 500)$

x	Exact Solution	Numerical Solution	Absolute Error
1.0000-	0.0000	0.0000	1.2709 e-18
0.6800-	0.4377	0.4361	0.0016
0.3600-	0.9744	0.9708	0.0036
0.0400	1.1680	1.1661	0.0019
0.3600	1.1740	1.1727	0.0013
0.7600	1.0826	1.0822	4.2924 e-04
1.0000	1.0000	1.0000	0.0000

Figure 1 illustrates the excellent agreement between the numerical solution and the exact solution across the entire domain.

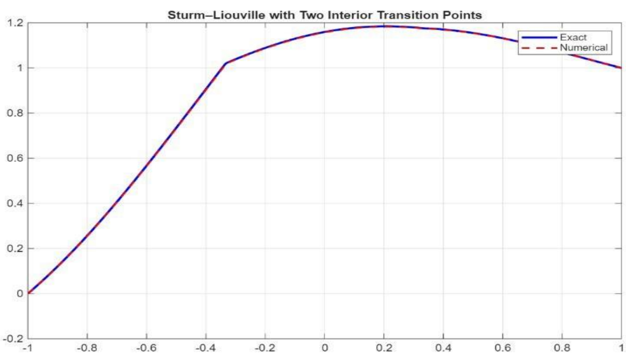

Figure 1. Comparison between the exact and numerical solutions for $N = 500$

A key feature of this problem is the discontinuity of the first derivative $y(x)$At the transition points. (Figure 2) shows the derivative of the numerical solution, clearly demonstrating the slope jumps at $x = \pm 1/3$, which confirms that our method correctly captures this essential behavior.

To verify the order of convergence, the maximum absolute error was calculated for different numbers of nodes N. As demonstrated in Table 2, doubling the number of nodes leads to a fourfold reduction in the maximum absolute error. The calculated convergence order p confirms that the method is indeed second-order accurate,$O(h^2)$, despite the presence of derivative discontinuities.

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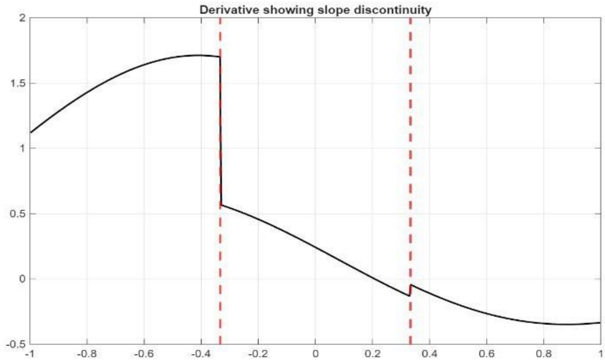

Figure 2. First derivative$y'(x)$ showing the slope jumps at the transition points $x = \pm 1/3$

The numerical stability observed in (Table 2) confirms that the specialized treatment of the two interior points does not introduce lower-order errors. The consistency between the calculated convergence order (p) and the theoretical expectations prove that the modified finite difference operators effectively bridge the solution across multiple interfaces while preserving the $O(h^2)$ precision.

Table 2. Numerical Convergence Results

N	h	Max Absolute Error	Convergence Order (p)
150	0.01333	× 8.7188 $^{3-}$10	—
300	0.00667	× 2.1797 $^{3-}$10	2.000
600	0.00333	× 5.4492 $^{4-}$10	2.000
1200	0.00167	× 1.3623 $^{4-}$10	2.000

Conclusion

In this study, the numerical framework of the Finite Difference Method (FDM) has been successfully extended to solve Sturm-Liouville problems featuring two interior transition points. This work generalizes the approach originally developed by Mukhtarov et al., transitioning from a single discontinuity to a multi-point interaction model.The numerical results demonstrate that doubling the number of nodes N Leads to a fourfold reduction in the maximum absolute error. Furthermore, the calculated convergence order of $p \approx 2.000$ confirms that the proposed treatment of multiple transmission conditions effectively preserves $O(h^2)$accuracy. This stability proves that the framework is robust and can be further generalized to handle n transition points without degrading the global precision of the solver.

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