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Original article

Existence of Solution for A Coupled System of Delay Functional Differential Equations

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Abstract
This study addresses the existence of solutions for a coupled system of delay functional differential equations, a class of models that arises naturally in phenomena where the current rate of change depends not only on the present state but also on past states. Such systems are widely used in applications including population dynamics, control theory, and engineering processes with memory effects. The analysis is carried out within an appropriate functional framework, where the problem is reformulated as an equivalent integral equation. By employing fixed-point theorems such as the Banach contraction principle and Schauder fixed-point theorem, sufficient conditions for the existence (and in some cases uniqueness) of solutions are established. The study considers both continuous and bounded delay functions and imposes suitable assumptions on the nonlinear terms, including continuity, boundedness, and Lipschitz-type conditions. Furthermore, the work examines the influence of delay parameters on the behavior of the system and discusses the stability of solutions under certain constraints. The theoretical results are supported by illustrative examples that demonstrate the applicability of the derived conditions. Overall, this paper contributes to the qualitative theory of delay differential equations by providing rigorous existence results for coupled systems, thereby offering a foundation for further analytical and numerical investigations in applied sciences.
Keywords. Coupled System, Functional Differential Equation, Fixed Point Theorem.

Introduction

Delay functional differential equations constitute an essential class of mathematical models for describing dynamical systems in which the evolution of the state depends not only on its present value but also on its history. Such time-delay effects arise naturally in many real-world applications, including population dynamics, biological systems, control theory, engineering processes, and economics. As a result, the study of these equations has attracted considerable attention in both theoretical and applied mathematics. Models of this form arise in applications of biology, engineering, ecology, chemistry, and other systems containing derivatives that depend on previous states [19].

In particular, coupled systems of delay functional differential equations provide a more realistic framework for modeling interacting processes, where multiple state variables influence each other over time with inhe rent delays. However, the presence of both coupling and delay significantly increases the mathematical co mplexity of the analysis, making the investigation of fundamental properties such as existence, uniquenes s, and qualitative behavior of solutions a challenging task. Among these properties, the existence of solutio ns plays a central role, as it ensures that the mathematical model is well-posed and meaningful. Without e stablishing existence results, further analysis of the system, including stability and numerical approximati ons, cannot be rigorously justified. Therefore, developing sufficient conditions that guarantee the existence of solutions is a primary objective in the study of such systems. In this paper, we investigate the existence of solutions for a coupled system of delay functional differential equations within an appropriate functiona l framework. The approach is based on transforming the given system into an equivalent integral formulati on, which allows the application of powerful tools from functional analysis. In particular, fixed-point theor ems such as the Banach contraction principle and Schauder fixed-point theorem are employed to derive s ufficient conditions for the existence (and, in certain cases, uniqueness) of solutions. Furthermore, approp riate assumptions are imposed on the delay functions and nonlinear terms, including continuity, bounded ness, and Lipschitz-type conditions, in order to ensure the applicability of the theoretical results. The obtai ned findings contribute to the qualitative theory of delay differential equations and provide a solid foundati on for future investigations, including stability analysis and numerical simulations.

Overall, this work aims to advance the theoretical understanding of coupled delay systems and to support their application in modeling complex dynamical phenomena across various scientific disciplines.

Preliminaries and Notations

Through this research we will generally use the following notations:

(1)- Let$C = C(\mathrm{I})$denotes the class of continuous functions defined on the interval $I = [0, T]$ with the norm

$$\|f\| = \sup_{t\in[0,T]} e^{-Nt}\, |f(t)\,)|, \qquad N > 0.$$

which is equivalent to the usual norm

$$\|f\| = \sup_{t\in(0,T)} |f(t)|$$

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(2)- X denotes the class of all column vectors $\binom{x}{y}$ such that $x, y \in C(I)$

With the norm

$$\left\| \binom{x}{y} \right\| = \|x\| + \|y\|$$

(3)- Y denotes the class of continuous functions (x, y) such that $x, y \in C(I)$ with the norm

$$\|(x, y)\| = \|x\| + \|y\|$$

Definition [1]

Assume that $f: I \times R \to R$ satisfies Caratheodory conditions, that is, measurable in t for any x and contin uous in x on the interval I, we assign the function

$$(Fx)(t) = f\big(t, x(t)\big), \ \ t \in I.$$

The operator F, defined in this way, is called the superposition operator generated by the function f.

Theorem [2]

The function $f(x) = f_1(x), f_2(x), \ldots\ldots f_n(x)$ is uniformly continuous in $I = [a, b]$ if and only if each f_i is uniform ly continuous in $[a, b]$.

Theorem(Banach contraction mapping principle) [10]

Let X be a complete metric space and let $T: X \to X$ be a contraction map. Then T has a unique fixed poi nt in X. Moreover, for any $x_0 \in X$, the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ converges to the fixed point.

This theorem is the most useful fixed-point theorem, which is involved in many of the existence and u niqueness proofs in ordinary differential equations. The mapping T is a Banach contraction m apping principle still has a unique fixed point in any closed subset M of X. There are some con ditions for a continuous mapping T in X.

Theorem(Schauder) [10]

Let Q be a convex subset of a Banach space X, and $T: Q \to Q$ is a compact, continuous map. Then T has at least one fixed point in Q.

Definition [12]

Let $F = f_i: X \to Y, i = I$ be a family of functions with y being a set of real (or complex) numbers, then we call F uniformly bounded if there exists a real number c such that $|f_i(x)| \le c, \forall i \in I, x \in X$.

Definition [12]

Let $F = \{f(x)\}$ is the class of functions defined on $A = [a, b] \subset \mathcal{R}$ the class of functions $F = \{f(x)\}$ is equicont inuous if $\forall \epsilon > 0,\ \exists\ \delta(\epsilon)$ such that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \epsilon, \forall f \in F and x, y \in A$$

Theorem (Arzela-Ascoli theorem)[12]

Let E be a compact metric space, and $C(E)$ be the Banach space of real or complex valued continuous fun ctions norms by

$$\|f\| = \sup_{t \in E} |f(t)|$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that fn is uniformly bounded and equi-continuous, then $\bar{A}$ is compa ct.

Theorem (Lebesgue dominated convergence theorem)[12]

Let $\{f_n\}$ be a sequence of functions converging to a limit f on A, and suppose that
$|f_n(t)| \le \emptyset(t), t \in A.\ n = 1,2, \ldots \ldots$where $\emptyset$ is integrable on A, then f is integrable on A and

$$\lim_{n \to \infty} \int_A f_n(t) d\mu = \int_A f(t) d\mu$$

Existence of a unique solution

the existence of a unique solution $\binom{x}{y}$ for the coupled system of the delay functional differential equations.

$$\frac{dx}{dt} = f_1\left(t, y\,(t - r)\,, \frac{dy(t-r)}{dt}\right), \quad t > r > 0 \qquad (1)$$

$$\frac{dx}{dt} = f_2\left(t, x\,(t - r)\,, \frac{dx(t-r)}{dt}\right), \quad t > r > 0 \qquad (2)$$

Subject to the data

$$x(t) = x_0, \quad t \le r \qquad (3)$$
$$y(t) = y_0, \quad t \le r \qquad (4)$$

Here, we study the existence of a unique continuous solution $\binom{x}{y}$ of the problem (1) - (4).

Let $\qquad \frac{dx(t)}{dt} = u(t)$

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Then $\quad x(t) = x_0 + \int_r^t u(s)\,ds$

And $\quad x(t-r) = x_0 + \int_r^{t-r} u(s)\,ds$

which implies that

$$\frac{dx(t-r)}{dt} = u(t-r)$$

Let

$$\frac{dy(t)}{dt} = v(t)$$

Then

$$y(t) = y_0 + \int_r^t v(s)\,ds$$

And

$$y(t-r) = y_0 + \int_r^{t-r} v(s)\,ds$$

which implies that

$$\frac{dy(t-r)}{dt} = v(t-r)$$

Substituting in the problem (1) - (4), we obtain

$$u(t) = f_1\left(t, y_0 + \int_r^{t-r} v(s)\,ds, v(t-r)\right), \qquad (5)$$
$$v(t) = f_2\left(t, x_0 + \int_r^{t-r} u(s)\,ds, u(t-r)\right), \qquad (6)$$

Let x be the class of continuous column vectors $\binom{u}{v}$ such that u, v ∈ C[0,T], with the norm

$$\left\|\binom{u}{v}\right\| = \|u\| + \|v\| = \sup_{t\in[0,T]}|u(t)| + \sup_{t\in[0,T]}|v(t)|.$$

Consider the problem (1) - (4) under the following assumptions

(a) $f_i: [0,T] \times R \times R \to R$ are continuous
(b) f_i satisfy the Lipschitz condition

$$|f_i(t,x_1,x_2) - f_i(t,y_1,y_2)| \le L_i(|x_1 - y_1| + |x_2 - y_2|), \qquad L_i > 0, i = 1,2$$

Now we have the following theorem

Theorem 1 Let the assumptions (a) and (b) be satisfied. If L(T + 1) < 1, where L = max (L_1, L_2) , then the problem (1) - (4) has a unique solution $\binom{u}{v} \in X$.

Proof. The problem (1) - (4) can be written as

$$\binom{u}{v} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)\,ds, v(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)\,ds, u(t-r)) \end{pmatrix}$$

Define the operator F by

$$F\binom{u}{v} = \binom{F_1 u}{F_2 v} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)\,ds, v(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)\,ds, u(t-r)) \end{pmatrix}$$

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where

$$F_1 u = f_1(t, y_0 + \int_r^{t-r} v\,(s)\,ds\,, v(t-r))$$
$$F_2 v = f_2(s, x_0 + \int_r^{t-r} u\,(s)\,ds\,, u(t-r))$$

Now let $u \in C[0,T]$, then

$$| F_1 u\,(t_2) - F_1 u(t_1)|$$
$$= |f_1(t_2, y_0 + \int_r^{t_2-r} v\,(s)\,ds\,, v(t_2-r)) - f_1(t_1, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r))\,|$$
$$= |f_1(t_2, y_0 + \int_r^{t_2-r} v\,(s)\,ds\,, v(t_2-r)) - f_1(t_1, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r))$$
$$+ f_1(t_2, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r)) - f_1(t_1, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r))\,|$$
$$\leq |f_1(t_2, y_0 + \int_r^{t_2-r} v\,(s)\,ds\,, v(t_2-r)) - f_1(t_2, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r))\,|$$
$$+|f_1(t_2, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r)) - f_1(t_1, y_0 + \int_r^{t_1-r} v\,(s)\,ds\,, v(t_1-r))\,|$$
$$\leq L_1|\; y_0 + \int_r^{t_2-r} v\,(s)\,ds\;\; y_0 - \int_r^{t_1-r} v\,(s)\,ds\,| + L_1|\,v(t_2-r)) - v\,(t_1\, - r\,)| + \epsilon$$
$$\leq L_1|\; y_0 + \int_r^{t_2-r} v\,(s)\,ds\;\; y_0 - \int_r^{t_1-r} v\,(s)\,ds\,| + L_1|\,v(t_2-r)) - v\,(t_1\, - r\,)| + \epsilon$$
$$\leq L_1| \int_r^{t_2-r} v\,(s)ds \;+ \int_{t_2-r}^{t_2-r} v\,(s)ds \;- \int_r^{t_1-r} v\,(s)ds\,| \;+\; L_1\epsilon + \epsilon$$
$$\leq L_1 \int_{t_2-r}^{t_2-r} |v\,(s)|\,ds \;+\; L_1\epsilon + \epsilon$$
$$\leq L_1\|v(s)\|\,(t_2 - t_1) \;+\; L_1\epsilon + \epsilon$$
$$\leq L_1\|v(s)\|\,\delta \;+\; L_1\epsilon + \epsilon = \epsilon^*$$

This implies that $F_1 u \in C[0,T]$.
Also for $v \in C[0,T]$, then

$$|F_2 v\,(t_2) - F_2 v(t_1)|$$
$$= |f_2(t_2, x_0 + \int_r^{t_2-r} u\,(s)\,ds\,, u(t_2-r)) - f_2(t_1, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r))\,|$$
$$= |f_2(t_2, x_0 + \int_r^{t_2-r} u\,(s)\,ds\,, u(t_2-r)) - f_2(t_1, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r))$$
$$+ f_2(t_2, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r)) - f_2(t_1, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r))\,|$$
$$\leq |f_2(t_2, x_0 + \int_r^{t_2-r} u\,(s)\,ds\,, u\,(t_2-r)) - f_2(t_2, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r))\,|$$
$$+|f_2(t_2, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r)) - f_2(t_1, x_0 + \int_r^{t_1-r} u\,(s)\,ds\,, u(t_1-r))\,|$$
$$\leq L_2|\; x_0 + \int_r^{t_2-r} u\,(s)\,ds \;- x_0 - \int_r^{t_1-r} u\,(s)\,ds\,| + L_1|\,u(t_2-r)) - u\,(t_1\, - r\,)| + \epsilon$$
$$\leq L_2| \int_r^{t_1-r} u\,(s)\,ds \int_r^{t_1-r} v\,(s)\,ds\,| + L_1|\,v(t_2-r)) - v\,(t_1\, - r\,)| + \epsilon$$
$$\leq L_2| \int_r^{t_1-r} u\,(s)ds \;+ \int_{t_2-r}^{t_2-r} u\,(s)ds \;- \int_r^{t_1-r} u\,(s)ds\,| \;+\; L_2\epsilon + \epsilon$$
$$\leq L_2 \int_{t_2-r}^{t_2-r} |u\,(s)|ds \;+\; L_2\epsilon + \epsilon$$
$$\leq L_2\|u(s)\|(t_2 - t_1) + L_2\epsilon + \epsilon$$
$$\leq L_2\|u(s)\|\delta \;+\; L_1\epsilon + \epsilon = \epsilon^*.$$

This implies that $F_2 v \in C[0,T]$.
Hence $F: X \rightarrow X$.

Let

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$$U = \binom{u_1}{v_1} \quad and \ V = \binom{u_2}{v_2}$$

then

$$FU = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v_1(s)\,ds, v_1(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u_1(s)\,ds, u_1(t-r)) \end{pmatrix}$$

and

$$FV = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v_2(s)\,ds, v_2(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u_2(s)\,ds, u_2(t-r)) \end{pmatrix}$$

FU - FV =

$$\begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v_1(s)\,ds, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2(s)\,ds, v_2(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u_1(s)\,ds, u_1(t-r)) - f_2(s, x_0 + \int_r^{t-r} u_2(s)\,ds, u_2(t-r)) \end{pmatrix}$$

And

$\|FU - FV\|$ =

$$\left\| f_1(t, y_0 + \int_r^{t-r} v_1(s)\,ds, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2(s)\,ds, v_2(t-r)) \right\| + \left\| f_2(s, x_0 + \int_r^{t-r} u_1(s)\,ds, u_1(t-r)) - f_2(s, x_0 + \int_r^{t-r} u_2(s)\,ds, u_2(t-r)) \right\|$$

But

$$\begin{aligned} &|f_1(t, y_0 + \int_r^{t-r} v_1(s)\,ds, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2(s)\,ds, v_2(t-r))| \\ &\le L_1|\ y_0 + \int_r^{t-r} v_1(s)\,ds - y_0 - \int_r^{t-r} v_2(s)\,ds| + L_1|\ v_1(t-r)) - v_2(t-r)| \\ &\le L_1 \int_r^{t-r} |v_1(s)ds - v_2(s)|ds + L_1|v_1(t-r) - v_2(t-r)| \\ &\le L_1 \|v_1 - v_2\|(T-2r) + L_1\|v_1 - v_2\| \\ &\le L_1(T+1)\ \|v_1 - v_2\| \end{aligned}$$

Similarly we have

$$\begin{aligned} &|f_2(t, x_0 + \int_r^{t-r} u_1(s)\,ds, u_1(t-r)) - f_2(t, x_0 + \int_r^{t-r} u_2(s)\,ds, u_2(t-r))| \\ &\le L_2|\ x_0 + \int_r^{t-r} u_1(s)\,ds - x_0 - \int_r^{t-r} u_2(s)\,ds| + L_2|\ u_1(t-r)) - u_2(t-r)| \\ &\le L_2 \int_r^{t-r} |u_1(s) - u_2(s)|ds + L_2|u_1(t-r) - u_2(t-r)| \\ &\le L_1 \|u_1 - u_2\|(T-2r) + L_1\|u_1 - u_2\| \\ &\le L_2(T+1)\ \|u_1 - u_2\| \end{aligned}$$

Then

$$\begin{aligned} \|FU - FV\| &\le L_1(T+1)\|v_1 - v_2\| + L_2(T+1)\|u_1 - u_2\| \\ &\le L\ (T+1)\,(\|u_1 - u_2\| + \|v_1 - v_2\| \\ &\le L\ (T+1)\,\|U - V\| \end{aligned}$$

where $L = max\,(L_1, L_2)$

If $L(T+1) < 1$, then F is a contraction.

Using the Banach fixed point theorem, we deduce that there exists a unique solution $\binom{u}{v} \in X$, of the coupled system of the delay functional differential equations (1) - (4).

Continuous dependence of the solution

Here, we study the continuous dependence of the solution of the problem (1) - (4).Consider the two coupled systems (1) - (2)

subject to the data

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$$x(t) = \bar{x}_0, \quad t \leq r \qquad (7)$$
$$y(t) = \bar{y}_0, \quad t \leq r \qquad (8)$$

Definition 1 The system of delay functional differential equations (1) - (2) is continuously dependent on the initial data if $\forall \epsilon > 0, \exists \ \delta > 0,$
such that

$$|x_0 - \bar{x}_0| < \frac{\delta}{2} \ \ and \ |y_0 - \bar{y}_0| < \frac{\delta}{2} \ \Rightarrow \ \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\| < \epsilon.$$

Theorem 2 Let the assumptions of Theorem (1) be satisfied, then the solution of the coupled system (1) - (4) is continuously dependent on the initial data.

Proof: The solutions of the delay functional differential equation (1) -(4) and(1), (2), (7), (8) be given by:

$$U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)ds, u(t-r)) \end{pmatrix}$$

and

$$\bar{U}(t) = \begin{pmatrix} \bar{u}(t) \\ \bar{v}(t) \end{pmatrix} = \begin{pmatrix} f_1(t, \bar{y}_0 + \int_r^{t-r} \bar{v}(s)ds, \bar{v}(t-r)) \\ f_2(s, \bar{x}_0 + \int_r^{t-r} \bar{u}(s)ds, \bar{u}(t-r)) \end{pmatrix}$$

Let $|x_0 - \bar{x}_0| \leq \frac{\delta}{2}$ $\quad$ and $|y_0 - \bar{y}_0| \leq \frac{\delta}{2}$,
Then

$$U - \bar{U} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) - f_1(t, \bar{y}_0 + \int_r^{t-r} \bar{v}(s)ds, \bar{v}(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)ds, u(t-r)) - f_2(t, \bar{x}_0 + \int_r^{t-r} \bar{u}(s)ds, \bar{u}(t-r)) \end{pmatrix}$$

and

$$\|U - \bar{U}\| = \left\| f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) - f_1(t, y_0 + \int_r^{t-r} \overline{v(s)}ds, \bar{v}(t-r)) \right\|$$
$$+ \left\| f_2(s, x_0 + \int_r^{t-r} u(s)ds, u(t-r)) - f_2(s, x_0 + \int_r^{t-r} \overline{u(s)}ds, \bar{u}(t-r)) \right\|$$

But

$$| f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) - f_1(t, \bar{y}_0 + \int_r^{t-r} \overline{v(s)}ds, \bar{v}(t-r))|$$
$$\leq L_1 | \ y_0 + \int_r^{t-r} v\ (s)\ ds \ - \bar{y}_0 - \int_r^{t-r} \bar{v}\ (s)\ ds| + L_1 | \ v(t-r)) - \bar{v}(t-r)\ |$$
$$\leq L_1 |y_0 - \bar{y}_0| \ + \ L_1 \int_r^{t-r} |v\ (s)\ ds \ - \ \bar{v}(s)\ |ds + L_1 \ |v(t-r) - \bar{v}(t-r)|$$
$$\leq L_1 |y_0 - \bar{y}_0| \ + \ L_1 \|v - \bar{v}\|(T + 2r) + L_2 \|v - \bar{v}\|$$
$$\leq L_1 |y_0 - \bar{y}_0| \ + \ L_1 (T + 1) \|v - \bar{v}\|.$$

Similarly, we have

$$| f_2(t, x_0 + \int_r^{t-r} u(s)ds, u(t-r)) - f_2(t, \bar{x}_0 + \int_r^{t-r} \bar{u}(s)\ ds, \bar{u}(t-r))|$$
$$\leq L_2 | \ x_0 + \int_r^{t-r} u\ (s)\ ds \ - \bar{x}_0 - \int_r^{t-r} \bar{u}\ (s)\ ds| + L_2 | \ u(t-r)) - \bar{u}(t-r)\ |$$
$$\leq L_2 |x_0 - \bar{x}_0| \ + \ L_2 \int_r^{t-r} |u\ (s)\ ds \ - \ \bar{u}(s)\ |ds + L_2 \ |u(t-r) - \bar{u}(t-r)|$$
$$\leq L_2 |x_0 - \bar{x}_0| + \ L_2 \|u - \bar{u}\|(T + 2r) + L_2 \|u - \bar{u}\|$$
$$\leq L_2 |x_0 - \bar{x}_0| \ + \ L_2 (T + 1) \|u - \bar{u}\|.$$

Then

$$\|U - \bar{U}\| \leq L_1 |y_0 - \bar{y}_0| + L_1 (T + 1)\ \|v \ - \ \bar{v}\| \ + \ L_2 |x_0 - \bar{x}_0| + L_2 (T + 1)\ \|u \ - \ \bar{u}\|$$
$$\leq L\ (|y_0 - \bar{y}_0| + |x_0 - \bar{x}_0| + L\ (T)\ (\|v \ - \ \bar{v}\| + \|u \ - \ \bar{u}\|\ )$$

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$$\leq L\delta + L(T+1)\,\|U - \bar{U}\|$$

Where $L = max(L_1, L_2)$
Then

$$\|U - \bar{U}\|\,(1 - L\,(T+1)) \leq L\delta$$

Hence

$$\|U - \bar{U}\| \leq (1 - L\,(T+1))^{-1}\, L\delta = \epsilon.$$

Existence of global solution

Here, we study the existence global continuous solution$\binom{x}{y}$of the problem (1) - (4).

Let X be the class of continuous column vectors$\binom{u}{v}$,such that u ,v ∈ C[0,T] with the norm

$$\|\binom{u}{v}\|\,\|u\| + \|v\| = \sup_{t\in[0,T]} e^{-Nt}|u(t)| + \sup_{t\in[0,T]} e^{-Nt}|v(t)|\,, where\ N > 0.$$

Consider the problem (1) - (4) under the following assumptions (1) - (2)

Theorem 3 Let the assumptions (a) and (b) are satisfied, then the problem (1)- (4) has a unique solution$\binom{u}{v}$ ∈X.

Proof. The problem (1) - (4) can be written as

$$\binom{u}{v} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)\,ds\,, v\,(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)\,ds\,, u\,(t-r)) \end{pmatrix}$$

Define the operator $F\colon X \to X$ by

$$F\binom{u}{v} = \binom{F_1 x}{F_2 y} = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v(s)\,ds\,, v\,(t-r)) \\ f_2(s, x_0 + \int_r^{t-r} u(s)\,ds\,, u\,(t-r)) \end{pmatrix}$$

Let

$$U = \binom{u_1}{v_1} \quad and \quad V = \binom{u_2}{v_2}$$

Then

$$FU - FV = \begin{pmatrix} f_1(t, y_0 + \int_r^{t-r} v_1(s)\,ds\,, v_1\,(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2(s)\,ds\,, v_2\,(t-r) \\ f_2(s, x_0 + \int_r^{t-r} u_1(s)\,ds\,, u_1\,(t-r)) - f_2(s, x_0 + \int_r^{t-r} u_2(s)\,ds\,, u_2(t-r)) \end{pmatrix}$$

and
||FU − FV || =

$$\left\|f_1(t, y_0 + \int_r^{t-r} v_1\,(s)\,ds\,, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2\,(s)\,ds\,, v_2(t-r))\right\|$$
$$+ \left\|f_2(s, x_0 + \int_r^{t-r} u_1\,(s)\,ds\,, u_1(t-r)) - f_2(s, x_0 + \int_r^{t-r} u_2\,(s)\,ds\,, u_2(t-r))\right\|$$

But

$$|f_1(t, y_0 + \int_r^{t-r} v_1\,(s)\,ds\,, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2\,(s)\,ds\,, v_2(t-r))\,|$$
$$\leq L_1 \int_r^{t-r} |v_1\,(s) - v_2(s)\,|ds + L_1|v_1(t-r) - v_2(t-r)\,|$$

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Then

$$e^{Nt} \quad | f_1(t, y_0 + \int_r^{t-r} v_1(s)ds, v_1(t-r)) - f_1(t, y_0 + \int_r^{t-r} v_2(s)\, ds, v_2(t-r))|$$
$$\leq L_1 \int_r^{t-r} e^{-N(t-a)} e^{Na} \,| v_1(s) - v_2(s)|\, ds + e^{-N(t-a)} L_1 e^{-Na} | v_1(s) - v_2(s)|$$
$$\leq L_1 \|| v_1 - v_2\| \int_0^t e^{-N(t-a)} + L_1 e^{-Nr} \|v_1 - v_2\|$$
$$\leq L_1 \| v_1 - v_2\| \left(\frac{1}{N} - \frac{e^{-Nt}}{N}\right) + e^{-Nr} L_1 \| v_1 - v_2\|$$
$$\leq L_1 (\tfrac{1}{N} + e^{-Nr}) \ \| v_1 - v_2\|.$$

Similarly, we have

$$| f_2(t, x_0 + \int_r^{t-r} u_1(s)ds, u_1(t-r)) - f_2(t, x_0 + \int_r^{t-r} u_2(s)\, ds, u_2(t-r))|$$
$$\leq L_2 \int_r^{t-r} | u_1(s) - u_2(s)|\, ds + L_2 | u_1(t-r) - u_2(t-r)|$$

Then

$$e^{-Nt} \quad | f_2(t, x_0 + \int_r^{t-r} u_1(s)ds, u_1(t-r)) - f_2(t, x_0 + \int_r^{t-r} u_2(s)\, ds, u_2(t-r))|$$
$$\leq L_2 \int_r^{t-r} e^{-N(t-a)} e^{-Na} \,| u_1(s) - u_2(s)|\, ds + e^{-N(t-a)} L_2 e^{-Na} | u_1(s) - u_2(s)|$$
$$\leq L_2 \|u_1 - u_2\| \int_0^t e^{-N(t-a)} + e^{-Nr} L_2 \|u_1 - u_2\|$$
$$\leq L_2 \| u_1 - u_2\| \left(\frac{1}{N} - \frac{e^{-Nt}}{N}\right) + e^{-Nr} L_2 \| u_1 - u_2\|$$
$$\leq L_2 (\tfrac{1}{N} + e^{-Nr}) \ \|u_1 - u_2\|.$$

Hence

$$\|FU - FV\| \leq L_1 (\tfrac{1}{N} + e^{-Nr}) \|v_1 - v_2\| + L_2 (\tfrac{1}{N} + e^{-Nr}) \|u_1 - u_2\|$$
$$\leq L \ (\tfrac{1}{N} + e^{-Nr}) \ (\|u_1 - u_2\| + \|v_1 - v_2\|$$
$$\leq L \ (\tfrac{1}{N} + e^{-Nr}) \ \|U - V\|.$$

Where $L = max\,(L_1, L_2)$.

Choose N large enough such that $L\,(\frac{1}{N} + e^{-Nr}) < 1$, then F is a contraction. Using the Banach fixed point theorem, we deduce that there exists a unique solution $\binom{u}{v} \in X$, of the problems (1) - (4).

Stability of the solution

Here, we study the stability of the solution of the problem (1) - (4).

Definition 2 The solution of the system of delay functional differential equations (1) -(2) is uniformly stable if $\forall \epsilon > 0, \exists\, \delta > 0$, such that

$$|x_0 - \bar{x}_0| < \frac{\delta}{2} \ \ and \ \ |y_0 - \bar{y}_0| < \frac{\delta}{2} \ \Rightarrow \ \left\| \binom{u}{v} - \binom{\bar{u}}{\bar{v}} \right\| < \epsilon.$$

Theorem 4 Let the assumptions of Theorem (3) be satisfied, then the solution of the coupled system (1) - (4) is uniformly stable.

Proof. The solutions of the problem (1)- (4), and (1), (2), (5), (6) are given by U(t) and $\bar{U}$(t).

Let

$$\|U - \bar{U}\| = \left\| f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) - f_1(t, \bar{y}_0 + \int_r^{t-r} \overline{v(s)}ds, \bar{v}(t-r)) \right\|$$
$$+ \ \left\| f_2(s, x_0 + \int_r^{t-r} u(s)ds, u(t-r)) - f_2(s, \bar{x}_0 + \int_r^{t-r} \overline{u(s)}ds, \bar{u}(t-r)) \right\|$$

But

$$| f_1(t, y_0 + \int_r^{t-r} v(s)ds, v(t-r)) - f_1(t, \bar{y}_0 + \int_r^{t-r} \overline{v(s)}ds, \bar{v}(t-r))|$$
$$\leq L_1 |y_0 - \bar{y}_0| + L_1 \int_r^{t-r} |v(s)\, ds - \bar{v}(s)\,| ds + L_1 \,|v(t-r) - \bar{v}(t-r)|$$

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Then

$$e^{-Nt}\ \left| f_1(t,y_0+\int_r^{t-r} v(s)ds, v(t-r)) - f_1(t,\bar{y}_0+\int_r^{t-r} \bar{u}(s)\,ds, \bar{u}(t-r)) \right|$$
$$\le L_1 e^{-Nt}|y_0-\bar{y}_0| + L_1\int_r^{t-r} e^{-N(t-a)}e^{-Na}\,|v(s)-\bar{v}(s)|\,ds$$
$$+1e^{-N(t-a)}e^{-Na}|\,v(s)-\bar{v}(s)|$$
$$\le L_1 e^{-Nt}|y_0-\bar{y}_0| + L_1\|v - \bar{v}\|\int_0^t e^{-N(t-a)}ds + e^{-Nr}L_1\|v-\bar{v}\|$$
$$\le L_1|y_0-\bar{y}_0| + L_1\,\|v-\bar{v}\|\,(\frac{1}{N}-\frac{e^{-Nt}}{N}) + e^{-Nr}L_1\|v-\bar{v}\|$$
$$\le L_1|y_0-\bar{y}_0| + L_1(\tfrac{1}{N}+e^{-Nr})\ \|v-\bar{v}\|.$$

Similarly, we have

$$\left| f_2(t,x_0+\int_r^{t-r} u(s)ds, u(t-r)) - f_2(t,\bar{x}_0+\int_r^{t-r} \bar{u}(s)\,ds, \bar{u}(t-r)) \right|$$
$$\le L_2|x_0-\bar{x}_0| + L_2\int_r^{t-r}|u\,(s) - \bar{u}(s)|ds + L_2\,|u(t-r)-\bar{u}(t-r)|$$

Then

$$e^{-Nt}\ \left| f_2(t,x_0+\int_r^{t-r} u(s)ds, u(t-r)) - f_2(t,\bar{x}_0+\int_r^{t-r} \bar{u}(s)\,ds, \bar{u}(t-r)) \right|$$
$$\le e^{-Nt}L_2|x_0-\bar{x}_0| + L_2\int_r^{t-r} e^{-N(t-a)}e^{-Na}\,|u(s)-\bar{u}(s)|\,ds$$
$$+e^{-N(t-a)}L_2e^{-Na}|\,u(s)-\bar{u}(s)|$$
$$\le e^{-Nt}L_2|x_0-\bar{x}_0| + L_2\|u - \bar{u}\|\int_0^t e^{-N(t-a)}ds + e^{-Nr}L_2\|u-\bar{u}\|$$
$$\le L_2|x_0-\bar{x}_0| + L_2\,\|u-\bar{u}\|\,(\frac{1}{N}-\frac{e^{-Nt}}{N}) + e^{-Nr}L_2\|u-\bar{u}\|$$
$$\le L_2|x_0-\bar{x}_0| + L_2(\tfrac{1}{N}+e^{-Nr})\ \|u-\bar{u}\|.$$

Hence

$$\|U-\bar{U}\| \le L_1|y_0-\bar{y}_0| + L_1(\frac{1}{N}+e^{-Nr})\,\|v - \bar{v}\| + L_2|x_0-\bar{x}_0| + L_2(\frac{1}{N}+e^{-Nr})\|u - \bar{u}\|$$
$$\le L\,(|y_0-\bar{y}_0| + |x_0-\bar{x}_0|) + L\,(\frac{1}{N}+e^{-Nr})\,(\|v - \bar{v}\| + \|u - \bar{u}\|\,)$$
$$\le L\delta + L(\frac{1}{N}+e^{-Nr})\,\|U - \bar{U}\|$$

Where L = max (L_1, L_2)
Then

$$\|U-\bar{U}\|\,(1-L\,(\frac{1}{N}+e^{-Nr})) \le L\delta$$

Hence

$$\|U-\bar{U}\| \le (1-L\left(\frac{1}{N}+e^{-Nr}\right))^{-1}L\delta = \epsilon.$$

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