

On Hypergroup Homomorphism

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Abstract

In this work, it has been proven that the kernel of a homomorphism between two hypergroups is not only a subhypergroup, but is precisely a normal subhypergroup, and the quotient of a hypergroup by that kernel is also a hypergroup. We use the kernel of a homomorphism between two groups to define hyper operation which make with that group a new hypergroup.

Keywords. Hyperoperation, Semihypergroup, Quasihypergroup, Hypergroup, Hypergroup Homomorphism.

Introduction

The theory of hyperstructures began in 1934 with F. Marty. In his presentation at the 8th congress of Scandinavian Mathematicians, the study of hyperstructures started seventy-five years ago with Marty's work, which first used a multiple-valued operator. Recall first the basic terms and definitions from the hyperstructure theory. As it is well known, Algebraic hyperstructures are generalizations of classical algebraic structures, In a classical algebraic structure, the binary operation of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the hyperoperation of two elements, is a subset of the same set, If this hyperoperation sends two elements to a singleton then the hyperoperation coincides with the classical binary operation. After introducing an algebraic hyperstructure, we focus on the study of homomorphisms between hyperstructures. We recall the basics of crisp homomorphism between hyperstructures, particularly between hypergroups. An algebraic hyperstructure is a non-empty set H together with a map $\oplus: A \times A \rightarrow \wp^*(A)$ called hyperoperation, where $\wp^*(A)$ denotes the set of all non-empty subsets of H . The couple (A, \oplus) is called a *hypergroupoid* [1]. If C and D are nonempty subsets of A , $x \in A$. We define: $C \oplus D = \bigcup_{c \in C, d \in D} c \oplus d$, $C \oplus x = C \oplus \{x\}$, $x \oplus D = \{x\} \oplus D$. A hyperstructure (H, \oplus) is called a *semihypergroup* if $\forall a, b, c \in H$; $(a \oplus b) \oplus c = a \oplus (b \oplus c)$, which means that:

$$\bigcup_{u \in b \oplus c} a \oplus u = \bigcup_{v \in a \oplus b} v \oplus c. \text{ If } x \in B \text{ and } C, D \text{ are nonempty subsets of } B, \text{ then}$$

$$C \oplus D = \bigcup_{c \in C, d \in D} c \oplus d, C \oplus x = C \oplus \{x\}, \text{ and } x \oplus D = \{x\} \oplus D.$$

Let A hypergroupoid (T, \oplus) is called a *quasi-hypergroup* if

$\forall a \in T$ we have $a \oplus T = T = T \oplus a$. This condition is also called the reproduction axiom.

Hypergroup

Definition 1.1 [1]

A hypergroupoid (G, \oplus) which is both a semihypergroup and a quasihypergroup is called a hypergroup.

Definition 1.2 [2]

A nonempty subset N of a hypergroup of G is called a subhypergroup of G if N itself is a hypergroup under the same hyperoperation as that of G . Hence, a nonempty subset N of a hypergroup (G, \oplus) is a subhypergroup if $\forall a \in N$, we have $a \oplus N = N \oplus a = N$.

Definition 1.3 [2]

A subhypergroup $A \subseteq G$ is called normal if $\forall x \in G$, we have $x \oplus A - x \subseteq A$, since $A \subseteq x \oplus A - x$. It follows:

$$A = x \oplus A - x, \forall x \in G.$$

Hyper Group Homomorphism

Definition 2.1. [3]

Let (G, \oplus) and (G', \oplus') be two hypergroups. A function $f: G \rightarrow G'$ is called:

• a **homomorphism (inclusion homomorphism)** if it satisfies the condition:

$$\forall x, y \in G, f(x \oplus y) \subseteq f(x) \oplus' f(y).$$

• f is a **good homomorphism** if; $\forall x, y \in G, f(x \oplus y) = f(x) \oplus' f(y)$.

• an **isomorphism** if it is a one-to-one and onto good homomorphism. If f is an isomorphism, then G and G' are said to be isomorphic ($G \cong G'$).

Let f be a homomorphism from hypergroup G into a hypergroup G' . Then, the set $\{x \in G: f(x) = 0\}$ is called the *kernel* ($\text{Ker } f$) of f .

Example 2.2.[4]

consider $G = \{0, 1, 2\}$ and $G' = \{a, b\}$ together with the following hyperoperations:

.	0	1	2
0	0	{0,1}	{0,2}
1	{0,1}	1	{1,2}
2	{0,2}	{1,2}	2

*	a	b
a	a	{a,b}
b	{a,b}	b

And suppose that $f : G \rightarrow G'$ is defined by $f(0) = f(1) = a$ and $f(2) = b$. Then, f is a good homomorphism.

Proposition 2.3.[4]

If φ is a homomorphism from a hypergroup G into a hypergroup G' , then

$$\varphi(-a) = -\varphi(a) \quad \forall a \in G.$$

Proof.

Because G is a hypergroup, $0 \in a - a$ for all $a \in G$, then we have $\varphi(0) \in \varphi(a) + \varphi(-a)$ or $0 \in \varphi(a) + \varphi(-a)$ which implies that $\varphi(-a) \in -\varphi(a) + 0$, therefore $\varphi(-a) = -\varphi(a)$ for all $a \in G$. \square

Lemma 2.4.

If $f : G \rightarrow G'$ is a hypergroup homomorphism, then the kernel $K = \ker(f)$ is a subhypergroup of G .

Proof.

Let $x, y \in K$. By definition of K , $f(x) = e_{G'}$ and $f(y) = e_{G'}$. Since f is a homomorphism, we have:

$$f(x \oplus y) \subseteq f(x) \oplus f(y), \text{ therefore } f(x \oplus y) \subseteq e_{G'} \oplus e_{G'}.$$

$e_{G'} \oplus e_{G'} = \{e_{G'}\}$, so $f(x \oplus y) \subseteq \{e_{G'}\}$. Thus, $f(z) = e_{G'}$ for all $z \in x \oplus y$, which implies $x \oplus y \subseteq K$. Therefore, K is closed under \oplus . Let $x \in K$, for any $y \in K$, we have $x \oplus y \subseteq K$. Now, $x \oplus K = \bigcup_{y \in K} (x \oplus y) \subseteq K$, also $K \oplus x = \bigcup_{y \in K} (y \oplus x) \subseteq K$. Since K is closed under \oplus , we also have. Thus, $x \oplus K = K \oplus x = K$. Hence, K is a subhypergroup of G .

Let $x, y \in K$. Since G is a hypergroup, the reversibility property holds for all $x, y \in G$. In particular, for any $z \in G$, $z \in x \oplus y \Rightarrow x \in z \ominus y$. Since $x, y \in K$ and K is closed under \oplus , $z \in K$. Thus, K inherits the reversibility property as a subset of G . \square

Theorem 2.5.

If $f : G \rightarrow G'$ is a Hypergroup homomorphism with kernel $K = \ker(f) = \{x \in G : f(x) = 0\}$, then the kernel is a normal subgroup of G .

Proof.

Since K is a subhypergroup of G (by Lemma). To show that K is normal, we verify $\forall x \in G, x \oplus K \ominus x \subseteq K$. Let $x \in G$ and $y \in K$. By definition of K , $f(y) = e_{G'}$. Consider any $z \in x \oplus y$. Since f is a homomorphism $f(z) \in f(x \oplus y) \subseteq f(x) \oplus f(y)$, therefore $f(z) \in f(x) \oplus e_{G'}$. By the identity property of G' , $f(x) \oplus e_{G'} = f(x)$. Thus, $f(z) \in f(x)$. Now consider $z \ominus x$. By the reversibility of \oplus in G , $z \ominus x \in K$ because $f(z \ominus x) \in f(z) \ominus f(x)$. Since $f(z) = f(x)$, we have $f(z \ominus x) \in f(x) \ominus f(x) = e_{G'}$, thus, $z \ominus x \in K$, therefore $x \oplus K \ominus x \subseteq K$. Hence, K is a normal subhypergroup of G . \square

Definition 2.6.[5]

Let f be a homomorphism from hypergroup G into hypergroup G' , let K be the kernel of f (K is a normal subhypergroup of G), the set of classes $G/K = \{aK \mid a \in G\}$ with the hyperoperation " \circ_K " as follows :

$$aK \circ_K bK = \bigcup \{ zK : z \in a \circ b \} \quad \forall a, b \in G.$$

which is called the *quotient hypergroup* of G by K .

Theorem 2.7.

If $f : G \rightarrow G'$ is a Hypergroup homomorphism with kernel $K = \ker(f) = \{x \in G : f(x) = 0\}$, then the quotient G/K is a hypergroup. where \circ_K is defined by $(aK) \circ_K (bK) = \bigcup_{x \in aK, y \in bK} (x \circ y)K, \forall aK, bK \in G/K$

Proof.

$$\begin{aligned} \text{Let } (aK \circ_K bK) \circ_K cK &= \bigcup \{ zK : z \in a \circ b \} \circ_K cK \\ &= \bigcup \{ wK : wK \in (a \circ b) \circ c \} \\ &= \bigcup \{ wK : w \in a \circ (b \circ c) \} \text{ since } a, b, c \in G \\ &= aK \circ_K \bigcup \{ uK : u \in b \circ c \} \\ &= aK \circ_K (bK \circ_K cK) \end{aligned}$$

Therefore, $(G/K, \circ_K)$ is a semihypergroup.

$$\begin{aligned} aK \circ_K G/K &= \bigcup \{ zK : z \in a \circ b \} \\ &= aK \circ_K \{ xK \mid x \in G \} \\ &= \bigcup \{ zK : z \in a \circ x \quad \forall x \in G \} \\ &= G/K. \end{aligned}$$

Similarly, $G/K \circ_K aK = G/K$

Therefore, $(G/K, \circ_K)$ is a quasihypergroup.

Hence $(G/K, \circ_K)$ is a hypergroup. \square

Connection between group and hypergroup

If H is a group, N is a normal subgroup of H, and \circ_N is the hyperoperation on H given by $x \circ_N y = xyN$. $\forall x, y \in H$. The next theorem shows that (H, \circ_N) is a hypergroup.

Theorem 3.1.

If H is a group with a normal subgroup N, then (H, \circ_N) is a hypergroup, where \circ_N is the hyperoperation defined as $x \circ_N y = xyN = \{xyn | n \in N\}$, is the left coset, $\forall x, y \in H$

Proof.

To show that the structure (H, \circ_N) is a semihypergroup, suppose that

$$\begin{aligned} (x \circ_N y) \circ_N z &= (xyN) \circ_N z \\ &= \bigcup_{n \in N} (xyn) \circ_N z \\ &= \bigcup_{n \in N} (xyn)zN \\ &= \bigcup_{n \in N} (xynzn) \text{ , since } nz=zn, N \text{ is a normal subgroup of } H \\ &= \bigcup_{n \in N} (xyznN) \\ &= xyzN. \end{aligned}$$

Consequently,

$$\begin{aligned} x \circ_N (y \circ_N z) &= x \circ_N (yzN) \\ &= \bigcup_{k \in N} x(yzn)N \\ &= xyzN. \end{aligned}$$

Thus, $(x \circ_N y) \circ_N z = x \circ_N (y \circ_N z)$. Hence, (H, \circ_N) is a Semihypergroup.

To show that the structure (H, \circ_N) is a quasihypergroup

if $\forall a \in H$, the following property holds: $a \circ_N H = H = H \circ_N a$, where $a \circ_N H = \{a \circ_N h | h \in H\}$ and $H \circ_N a = \{h \circ_N a | h \in H\}$. $\forall a \in H$, applying the hyperoperation \circ_N between a and any element of H results in the entire set H. Suppose that $a \circ_N H = \bigcup_{h \in H} (a \circ_N h) = H$, $H \circ_N a = \bigcup_{h \in H} (h \circ_N a) = H$.

This means that the hyperoperation \circ_N is "compatible" with the group structure of H. \square

Example 3.2.

Let $H = \{1, -1, i, -i\}$, where $i = \sqrt{-1}$, (H, \cdot) is a group, and the operation. is defined as the table:

.	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Clearly, $N = \{1, -1\}$ is a normal subgroup of H.

$H/N = \{N, iN\}$ we can defined \circ_N as following: $x \circ_N y = xyN$. $\forall x, y \in H$.

We can see clearly from the next table that (H, \circ_N) is a hypergroup

\circ_N	1	-1	i	-i
1	N	N	iN	iN
-1	N	N	iN	iN
i	iN	iN	N	N
-i	iN	iN	N	N

We know that any kernel of a homomorphism is a normal subgroup, the next Corollary is proved by Theorem 3.1.

Corollary 3.3.

If $f: H \rightarrow H'$ is a group homomorphism with kernel $K = \ker(f) = \{x \in H | f(x) = 0_{H'}\}$, then (H, \circ_K) is a hypergroup, where \circ_K is the hyperoperation defined as $x \circ_K y = xyK$; $\forall x, y \in H$. Here is an example of a group homomorphism $f: H \rightarrow H'$, where $K = \ker(f)$ is a normal subgroup, and the resulting hypergroup (H, \circ_K) satisfies the given properties.

Example 3.4.

Define the homomorphism $f: Z \rightarrow Z_2$ by $f(x) = [x]_2$. The kernel of f is:

$K = \ker f = \{x \in Z | f(x) = 0\} = \{x \in Z | [x]_2 = [0]_2\}$. Thus, $K = 2Z$, the set of all even integers. Clearly, K is a normal subgroup of Z because Z is abelian.

$\forall x, y \in Z$, the hyperoperation \circ_K is defined as: $x \circ_K y = x+y+K = \{x+y+k | k \in K\}$.

Since $K = 2Z$, we can write:

$$x \circ_K y = \{x+y+2n | n \in Z\}.$$

This represents the set of all integers of the form $x+y+2n$, i.e., all integers congruent to $[x+y]_2$.

To check Semihypergroup, let $x, y, z \in Z$. Using the definition of \circ_K :

$$(x \circ_K y) \circ_K z = (x+y+K) \circ_K z.$$

Expanding $x + y + K$, we get:

$$(x \circ_K y) \circ_K z = \bigcup_{k \in K} (x + y + k) \circ_K z = \bigcup_{k \in K} ((x + y + k) + z + K).$$

Since $k \in K = 2Z$, the result is:

$$(x \circ_K y) \circ_K z = x + y + z + K.$$

On the other hand, $x \circ_K (y \circ_K z) = x \circ_K (y+z+K)$.

Expanding $y+z+K$, we get:

$$x \circ_K (y \circ_K z) = \bigcup_{k \in K} x \circ_K (y+z+K) = \bigcup_{k \in K} (x + (y + z + k) + K).$$

This simplifies to $x \circ_K (y \circ_K z) = x+y+z+K$. Thus $(x \circ_K y) \circ_K z = x \circ_K (y \circ_K z)$,

We need to verify Quasihypergroup $\forall a \in Z$:

For $a \circ_K Z$, we calculate:

$$a \circ_K Z = \bigcup_{b \in Z} (a \circ_K b).$$

Since $a \circ_K b = a+b+K$, we have $a \circ_K Z = \bigcup_{b \in Z} (a + b + K)$. This is the set of all integers, because $a+b+K$ covers all congruence classes modulo 2, $\forall b \in Z$, and Z is closed under addition. Similarly, for $Z \circ_K a$, we calculate $Z \circ_K a = \bigcup_{b \in Z} Z(b \circ_K a)$. Since $b \circ_K a = b+a+K$, we have $Z \circ_K a = \bigcup_{b \in Z} (b + a + K)$

Again, this is the set of all integers, because $b+a+K$ covers all congruence classes modulo 2, $\forall b \in Z$. Thus:
 $a \circ_K Z = Z = Z \circ_K a$.

Conclusion

In the context of hypergroups, the kernel of a homomorphism between two hypergroups G and G' is defined as a normal subhypergroup of G . This property ensures that the quotient structure G/K , formed by factoring G over the kernel K , itself constitutes a hypergroup. Building on this foundation, one can utilize the kernel of a homomorphism between two groups to define a hyperoperation. This hyperoperation, when applied to the group, generates a new hypergroup structure, thereby extending classical group theory into the broader framework of hypergroup theory.

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