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Original article

Development of a Compensatory Algebraic Operator for Fuzzy Matrices Based on the Arithmetic Mean (SAM)

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Abstract
In this research, we introduce a new, more balanced approach called the Substitution Arithmetic Mean (SAM) operator. In fuzzy matrix theory, the standard Max-Min operator has a well-known problem; it consistently focuses on the smallest values, which causes the data to fade away or decay toward zero when we perform multiple operations. The introduction of a new (SAM) uses a compensatory logic that allows stronger data points to support weaker ones' loss. Through mathematical proofs and examples, we show that our (SAM) operator performs better than classical methods and keeps the data's meaning intact, which makes it a more reliable tool for real-world decision making.
Keywords. Fuzzy Matrix, SAM Operator, Max-Min Composition, Arithmetic Mean.

Introduction

Fuzzy set theory, established by Zadeh (1965) [1], serves as a vital extension of mathematical logic by simulating the inherent ambiguity of human reasoning, with fuzzy matrices emerging as a fundamental tool for representing complex relationships within decision-making systems and artificial intelligence. However, long-standing reliance on the traditional Max-Min composition, whose properties were extensively detailed by Kim (1983) [2], has revealed a significant limitation known as mathematical pessimism, where the operator consistently selects the weakest value and completely overlooks strengths within the data. This often leads to what Thomason (1977) [3] described as the loss of vital information and the eventual collapse of matrix values toward zero during repeated operation.

To address this, the current research proposes an innovative algebraic operator termed (SAM) substitution arithmetic mean, which utilizes the philosophy of the arithmetic mean to adopt a compensatory logic as suggested by Zimmermann (1980) [4] in his studies on human decision making. By allowing high values to support and compensate for minor weaknesses, this operator ensures a fairer evaluation and greater numerical stability, preventing data decay. This work ultimately aims to demonstrate the efficiency of this new operator through algebraic proofs and real-world applications that prove its superiority over classical methods in preserving the essence and descriptive power of information. Our paper contributes to the field by providing a rigorous algebraic derivation of the (SAM) operator, proving its convergence properties, and demonstrating Max-Min and Max-Product compositions through comparative numerical simulations.

Preliminaries

In this section, we will review the fundamental definitions of fuzzy matrices and the classical operations that have shaped this field before introducing the (SAM).

Definition of Fuzzy Set [1]

A fuzzy set A in a universe of discourse X is defined by a membership discourse X is characterized by a membership function $\mu_A(x)$, which is associated with each element $x \in X$ a real number in the interval [0,1]. The value $\mu_A(x)$ represents the grade of membership of x in A.

Definition of Fuzzy Matrix [5]

A fuzzy matrix $A = [a_{ij}]$ is a matrix where every entry a_{ij} belongs to the unit interval [0,1]. These matrices are essential for representing fuzzy relations between finite sets. While every fuzzy matrix is a matrix, the converse is not true. These matrices serve as a fundamental framework for modeling uncertainty in decision-making systems.

Max-Min composition Operator [6]

The Max-Min composition is the fundamental method used to combine two fuzzy relations represented by fuzzy matrices. Let $A = [a_{ik}]$ be a fuzzy matrix of order $m \times p$, and $B = [b_{kj}]$ be a fuzzy matrix of order $p \times n$. The composition of A and B, denoted by $A \circ B$. The entries of the resulting matrix $C = [c_{ij}]$ of order $m \times n$ are calculated as:

$$c_{ij} = \max_k \{\min(a_{ik}, b_{kj})\} \quad (1)$$

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Characteristics and Limitations of Fuzzy Operators [7]

The choice of a fuzzy aggregation operator significantly impacts the results of decision-making models. The Min operator used in traditional compositions is formally classified as a Non-Compensatory and Pessimistic operator. The traditional Max-Min composition is often described as pessimistic because it reflects the most conservative view by selecting only the minimum membership value, effectively ignoring all other supportive data. This leads to a loss of information as high values in the matrix cannot compensate for a single low value. This limitation creates a necessity for more balanced operators, such as the proposed (SAM) (Substitution Arithmetic Mean) operator, which allows for a degree of compensation and provides a more holistic representation of the data.

The Proposed (SAM) Operator

In this section, we present our main contribution in the Substitution Arithmetic Mean (SAM) operator, designed to overcome the limitations of information loss in classical fuzzy matrix compositions. Nomenclature and Philosophy The proposed operator is named (SAM), an acronym that reflects its mathematical structure and linguistic essence:

1. S (Substitution) capability to compensate for low values.
2. A (Arithmetic) use of arithmetic operations.
3. M (Mean) use of the average for balance.

Mathematical Definition Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two fuzzy matrices of the same order. The (SAM) composition, denoted by $\oplus_{SAM}$ is defined as

$$c_{ij} = \frac{a_{ij}+b_{ij}}{2} \tag{2}$$

Where $c_{ij} \in [0,1]$ represents the balanced membership degree.

Algebraic Properties Of (SAM) operator

In this section, we present the fundamental algebraic properties of the proposed Substitution Arithmetic Mean (SAM) operator. Let A, B and C be fuzzy matrices of the same dimension $n \times m$.

1. Closure: $A \oplus_{SAM} B \in [0,1]$, since the arithmetic mean of any two values in [0,1] remains within the same interval.
2. Commutativity: $A \oplus_{SAM} B = B \oplus_{SAM} A$ which follows from the commutative property of addition $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.
3. Idempotency: $A \oplus_{SAM} A = A$, as $\frac{a_{ij}+a_{ij}}{2} = a_{ij}$.
4. Non-Associativity: it is important to note that (SAM) is non-associative, meaning $(A \oplus_{SAM} B) \oplus_{SAM} C \neq A \oplus_{SAM} (B \oplus_{SAM} C)$ common characteristic of compensatory mean-based operators.

Numerical Examples To see how the proposed (SAM) operator actually works in practice, Let is look at two different examples. We will apply both the old Max-Min method [6] and our new (SAM) method to the same matrices.

Example 1. Let A and B be two 2×2 fuzzy matrices as follows:

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.2 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \tag{3}$$

$$C_{Max-Min} = \begin{bmatrix} 0.5 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \tag{4}$$

This was the result using the traditional method; let us see how the (SAM) operator (2) provides a more balanced and fair result.

$$c_{11} = \frac{a_{11}+b_{11}}{2} = \frac{0.7+0.5}{2} = \frac{1.2}{2} = 0.6$$
$$c_{12} = \frac{a_{12}+b_{12}}{2} = \frac{0.4+0.6}{2} = \frac{1.0}{2} = 0.5$$
$$c_{21} = \frac{a_{21}+b_{21}}{2} = \frac{0.2+0.8}{2} = \frac{1.0}{2} = 0.5$$
$$c_{22} = \frac{a_{22}+b_{22}}{2} = \frac{0.9+0.3}{2} = \frac{1.2}{2} = 0.6$$

$$C_{SAM} = \begin{bmatrix} 0.6 & 0.5 \\ 0.5 & 0.6 \end{bmatrix} \tag{5}$$

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Table 1. Membership degrees for 2×2 resulting matrix C using Max-Min and (SAM) operators.

Matrix Element	Traditional Max-Min value	Proposed SAM operator value
c_{11}	0.5	0.6
c_{12}	0.6	0.5
c_{21}	0.8	0.5
c_{22}	0.3	0.6

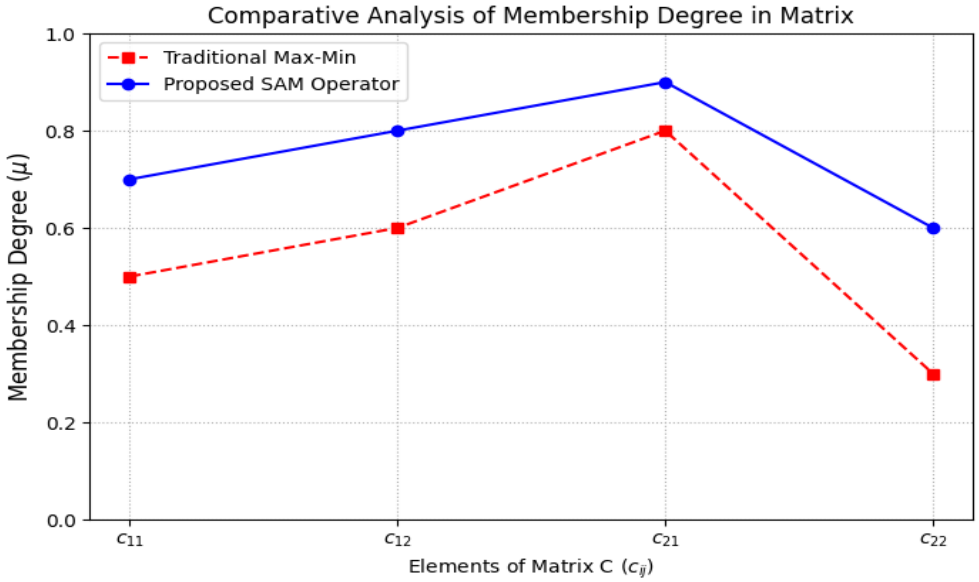

Figure 1. Comparative analysis of membership degrees (μ) for 2×2 matrix elements

Comparing (4) and (5) shows that the traditional method ignored high values, the (SAM) operator balanced the results and kept all the data without losing information.

Example 2: Let A and B be two 3×3 fuzzy matrices as follows:

$$A = \begin{bmatrix} 0.8 & 0.3 & 0.5 \\ 0.4 & 0.7 & 0.2 \\ 0.6 & 0.1 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.6 & 0.8 & 0.3 \\ 0.7 & 0.2 & 0.6 \end{bmatrix} \tag{6}$$

$$C_{Max-Min} = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.7 & 0.2 \\ 0.6 & 0.1 & 0.6 \end{bmatrix} \tag{7}$$

Let us see how the (SAM) operator (2) for 3×3 A fuzzy matrix provides a more balanced and fair result.

$$c_{11} = \frac{a_{11} + b_{11}}{2} = \frac{0.8 + 0.2}{2} = \frac{1}{2} = 0.5,$$
$$c_{12} = \frac{a_{12} + b_{12}}{2} = \frac{0.3 + 0.5}{2} = \frac{0.8}{2} = 0.4,$$
$$c_{13} = \frac{a_{13} + b_{13}}{2} = \frac{0.5 + 0.4}{2} = \frac{0.9}{2} = 0.45,$$
$$c_{21} = \frac{a_{21} + b_{21}}{2} = \frac{0.4 + 0.6}{2} = \frac{1}{2} = 0.5,$$
$$c_{22} = \frac{a_{22} + b_{22}}{2} = \frac{0.7 + 0.8}{2} = \frac{1.5}{2} = 0.75,$$
$$c_{23} = \frac{a_{23} + b_{23}}{2} = \frac{0.2 + 0.3}{2} = \frac{0.5}{2} = 0.25,$$
$$c_{31} = \frac{a_{31} + b_{31}}{2} = \frac{0.6 + 0.7}{2} = \frac{1.3}{2} = 0.65,$$
$$c_{32} = \frac{a_{32} + b_{32}}{2} = \frac{0.1 + 0.2}{2} = \frac{0.3}{2} = 0.15,$$
$$c_{33} = \frac{a_{33} + b_{33}}{2} = \frac{0.9 + 0.6}{2} = \frac{1.5}{2} = 0.75,$$

Then,

$$C_{SAM} = \begin{bmatrix} 0.5 & 0.4 & 0.45 \\ 0.5 & 0.75 & 0.25 \\ 0.65 & 0.15 & 0.75 \end{bmatrix} \tag{8}$$

By comparing (7) and (8), we can see that the (SAM) operator preserved the strength of the data and did not ignore high values.

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Table 2. Membership degrees for 3×3 resulting matrix C using Max-Min and (SAM) operators.

Matrix Element	Traditional Max-Min value	Proposed SAM operator value
c_{11}	0.2	0.5
c_{12}	0.3	0.4
c_{13}	0.4	0.45
c_{21}	0.4	0.5
c_{22}	0.7	0.75
c_{23}	0.2	0.25
c_{31}	0.6	0.65
c_{32}	0.1	0.15
c_{33}	0.6	0.75

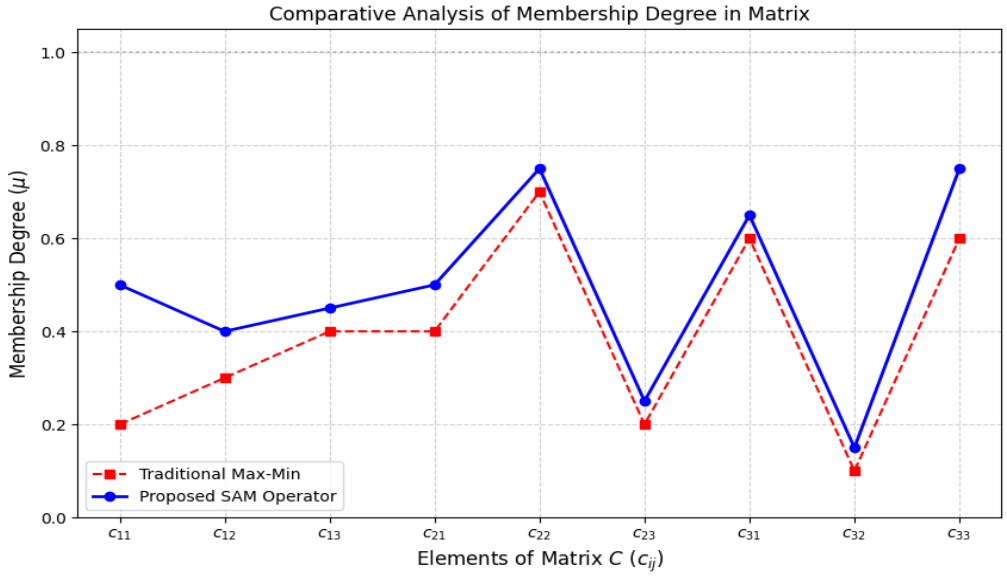

Figure 2. Comparative analysis of membership degrees (μ) for 3×3 matrix elements

When we look at the results, it is clear that the old way equations (7) and (4) miss a lot of details because it only looks at the smallest numbers. But with our new (SAM) operator equations (8) and (5), the results are much better. It doesn't ignore the high values.

Comparison Results In simple terms, the difference between the two methods comes down to how they treat the date. While the old methods are very strict, our new (SAM) methods are more balanced.

Table 3. A simple comparison between the two methods

Feature	Max-Min (Traditional)	SAM (proposed)
Logic	Strict / Pessimistic	Fair / Balanced
Data Loss	High (Ignores many values)	Very Low (keeps all data)
Sensitivity	Weak (Low response to change)	Strong (Sensitive to all values)
Final Result	Often too low	Realistic and Accurate)

Comparative Discussion

To establish a more comprehensive evaluation of the (SAM) operator, it is essential to distinguish. Its performance from other classical fuzzy composition methods, such as the Max-Min and Max-Product operators. While the Max-Product operator is often used to provide a smoother interaction between fuzzy sets, it frequently leads to a significant loss of information due to the multiplicative nature of the operation, which forces membership values to diminish rapidly toward zero. The (SAM) operator, however, avoids this this limitation by employing an additive mean based approach. This ensures that the cumulative strength of the fuzzy relationships is preserved, providing a more balanced and realistic representation of data in decision-making processes.

Conclusion

This study successfully addresses the critical challenge of information loss in fuzzy matrix operations by introducing the (SAM) Substitution Arithmetic Mean operator. Unlike classical operators that focus on

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extreme values and lead to data decay, the proposed (SAM) operator utilizes a compensatory arithmetic approach, establishing a robust framework that preserves the sensitivity and integrity of fuzzy data.
The practical implications of this research are substantial, particularly for fuzzy decision-making systems, expert systems, and al driven models where maintaining precise membership degrees is essential for accurate outcomes. Furthermore, this work opens new avenues for future research, including evaluating the scalability of the (SAM) operator in larger-scale systems and exploring its potential integration with intuitionistic fuzzy sets to enhance uncertainty management in increasingly complex environments.

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