Original article

The Connection between Topological and Algebraic Equivalence on Topological Hypergroups

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Abstract
Based on the algebraic and topological structure of topological hypergroups, an important question arises. Does topological equivalence achieve algebraic equivalence and its converse?. The purpose of this paper is to show that this claim is false in general. By counterexample, we construct two finite topological hypergroups and define a bijective algebraic isomorphism between them that fails to be a homeomorphism. This example shows that additional conditions are required to establish any equivalence between algebraic and topological isomorphisms in topological hypergroups.
Keywords. Hypergroup, Topological Group, Topological Hypergroup, Isomorphism of Topological Hypergroup.

Introduction

The French mathematician Marty proposed the idea of hyperstructures, and particularly the idea of a hypergroup, which is a generalization of classical algebraic structures. In classical algebraic structures, the composition of two elements results in a single element, whereas in hyperstructures, the composition of two elements produces a non-empty set of elements [1]. More exactly, let $\mathcal{H}$ be a nonempty set and let $\mathcal{P}^*(\mathcal{H})$ be the set of all nonempty subsets of $\mathcal{H}$. Then a hyperoperation on H is a map $\circ: \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{P}^*(\mathcal{H})$ and the couple $(\mathcal{H}, \circ)$ is called a hypergroupoid [2]. A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of $\mathcal{H}$, we have $\mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c}) = (\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c}$ [1]. A hypergroupoid $(\mathcal{H}, \circ)$ is called a quasi-hypergroup if for all $\mathfrak{a}$ of $\mathcal{H}$ we have $\mathfrak{a} \circ \mathcal{H} = \mathcal{H} \circ \mathfrak{a} = \mathcal{H}$. This condition is also called the reproduction axiom [3]. A topological group is a set endowed with two structures, namely that of a topological space and that of a group. The algebraic properties of the group affect the topological properties of the space, and vice versa. Let $(\mathcal{G}, \circ)$ be a group and $\mathcal{G}$ be a topological space such that the mappings $\mathfrak{a}, \mathfrak{b} \rightarrow \mathfrak{a} \circ \mathfrak{b}$ from $\mathcal{G} \times \mathcal{G}$ to G and $\mathfrak{a} \rightarrow \mathfrak{b}^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ are continuous. In this definition, $\mathcal{G} \times \mathcal{G}$ has the product topology [4]. A topological hypergroup is a generalization of topological groups. The hypergroup is endowed with a topology such that these hyperoperations are continuous. In addition, a topology is induced on the collection of subsets arising from the hyperoperation in a way that is consistent with the original topology [4].

Preliminaries

Definition 2.1.[5]

A hypergroupoid $(\mathcal{H}, \circ)$ which is both a semi-hypergroup and a quasi-hypergroup, is called a *hypergroup*.

Example 2.2. Consider $\mathcal{H}_1 = \{e, a, b\}$, where the hyperoperation $\circ$ define as follows:

$\circ$	e	a	b
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\mathcal{H}_1$

Clearly associativity holds, for instance. For any $e, a, b \in \mathcal{H}_1 \Longrightarrow e \circ (a \circ b) = e \circ \{b\} = \{b\}$ $(e \circ a) \circ b = \{a\} \circ b = \{b\}$. Then, $(\mathcal{H}_1, \circ)$ is a semihypergroup. $(\mathcal{H}_1, \circ)$ Is quasihypergroup indeed, for any $e, a, b \in \mathcal{H}_1 \Longrightarrow \{a\} \circ \mathcal{H}_1 = \{a\} \circ \{e, a, b\} = \{a\} \cup \{e, a\} \cup \{b\} = \mathcal{H}_1$. $\mathcal{H}_1 \circ \{a\} = \{e, a, b\} \circ \{a\} = \{a\} \cup \{e, a\} \cup \{b\} = \mathcal{H}_1$. Then, $(\mathcal{H}_1, \circ)$ is a hypergroup.

Example 2.3. Consider $\mathcal{H}_2 = \{u, v, w\}$, where the hyperoperation $\acute{\circ}$ define as follows:

$\acute{\circ}$	u	v	w
u	$\{u\}$	$\{v\}$	$\{w\}$
v	$\{v\}$	$\{u, v\}$	$\mathcal{H}_2$
w	$\{w\}$	$\mathcal{H}_2$	$\{u, v\}$

Clearly, associativity holds, for instance. For any $u, v, w \in \mathcal{H}_2 \Longrightarrow v \acute{\circ} (u \acute{\circ} w) = v \acute{\circ} \{w\} = \mathcal{H}_2$ $(v \acute{\circ} u) \acute{\circ} w = \{v\} \acute{\circ} w = \mathcal{H}_2$. Then, $(\mathcal{H}_2, \acute{\circ})$ is semi hypergroup. A semi-hypergroup $(\mathcal{H}_2, \acute{\circ})$ is a quasihypergroup. For any $u, v, w \in \mathcal{H}_2 \Longrightarrow \{u\} \acute{\circ} \mathcal{H}_2 = \{u\} \acute{\circ} \{u, v, w\} = \{u\} \cup \{v\} \cup \{w\} = \mathcal{H}_2$. $\mathcal{H}_2 \acute{\circ} \{u\} = \{u, v, w\} \acute{\circ} \{u\} = \{u\} \cup \{v\} \cup \{w\} = \mathcal{H}_2$.
Then, $(\mathcal{H}_2, \acute{\circ})$ is a hypergroup.

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Results

Let $(\mathcal{H},\mathfrak{I})$ be a topological space. Then, the family $\mathfrak{B}$ consisting of all sets

$$\mathcal{S}_{\mathcal{V}} = \{\mathcal{U} \in \mathcal{P}^*(\mathcal{H}) : \mathcal{U} \subseteq \mathcal{V} ; \mathcal{V} \in \mathfrak{I}\}$$

Is a base for a topology on $\mathcal{P}^*(\mathcal{H})$. This topology is denoted $\mathfrak{I}^*$. Let $(\mathcal{H},\mathfrak{I})$ be a topological space. Then, we consider the product topology on $\mathcal{H} \times \mathcal{H}$ and the topology $\mathfrak{I}^*$ on $\mathcal{P}^*(\mathcal{H})$ [4].

Definition 3.1. [6]

Let $(\mathcal{H},\mathfrak{I})$ be a topological space and $(\mathcal{H},\circ)$ a hypergroup. Define $\mathfrak{I}^*$ as a topology on the set of nonempty subsets $\mathcal{P}^*(\mathcal{H})$, $(\mathcal{H},\circ,\mathfrak{I})$ is a *topological hypergroup* if the following two maps $\mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$

$$(x,y) \mapsto x \circ y, \qquad (x,y) \mapsto x/y = \{z \in \mathcal{H}: x \in z \circ y\}.$$

are continuous with respect to the product topology on $\mathcal{H} \times \mathcal{H}$ and $\mathfrak{I}^*$ on $\mathcal{P}^*(\mathcal{H})$. Therefore, the mapping $\circ$ is continuous.

Example 3.2.

By Example 2.2, Define $\mathfrak{I}$ is a topology on a hypergroup $(\mathcal{H}_1,\circ)$ as following:

$$\mathfrak{I} = \{\phi ,\mathcal{H}_1,\{e\} \}.$$

We know that $\mathcal{P}^*(\mathcal{H}_1) = \{\{e\},\{a\},\{b\},\{e,a\},\{e,b\} ,\{a,c\},\mathcal{H}_1 \}$.

Now, Define base for a topology on $\mathcal{P}^*(\mathcal{H}_1)$, $\mathcal{S}_{\emptyset} = \{\emptyset\}$, $\mathcal{S}_{\{\mathcal{H}\}} = \mathcal{P}^*(\mathcal{H}_1)$

$$\mathcal{S}_{\{b\}} = \{\{e\}\} \Longrightarrow \mathfrak{I}^* = \Big\{\emptyset, \mathcal{P}^*(\mathcal{H}_1), \{\{e\}\}\Big\}.$$

Now, to find $\mathcal{H}_1 \times \mathcal{H}_1$ and $\mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}$.

$$\mathcal{H}_1 \times \mathcal{H}_1 = \{(e,e),(e,a),(e,b),(a,e),(a,a),(a,b),(b,e),(b,a),(b,b)\}.$$

$$\mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1} = \{\emptyset ,\mathcal{H}_1 \times \mathcal{H}_1,\{(e,e)\},\{(e,e),(a,e),(b,e)\},\{(e,e),(e,a),(e,b)\}$$
$$\{(e,e),(a,e),(b,e),(e,a),(e,b)\}\}.$$

$$\circ^{-1}(\emptyset) = \emptyset \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}.$$
$$\circ^{-1}(\mathcal{P}^*(\mathcal{H}_1)) = \mathcal{H}_1 \times \mathcal{H}_1 \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}$$
$$\circ^{-1}(\{e\}) = \{(e,e\} \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}$$

Hince $\circ$ is continuous.

By Definition 3.1, we define $/: \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{P}^*(\mathcal{H}_1)$ as follows:

/	**e**	**a**	**b**
e	$\{e\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{e,a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\mathcal{H}_1$

$$/^{-1}(\emptyset) = \emptyset \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}.$$
$$/^{-1}(\mathcal{P}^*(\mathcal{H}_1)) = \mathcal{H}_1 \times \mathcal{H}_1 \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}$$
$$/^{-1}(\{e\}) = \{(e,e)\} \in \mathfrak{I}_{\mathcal{H}_1\times\mathcal{H}_1}$$

Hince / is continuous. Therefore $(\mathcal{H}_1,\circ,\mathfrak{I})$ is a topological hypergroup.

Example 3.3. By Example 2.3. Define $\acute{\mathfrak{I}}$ is a topology on a hypergroup $(\mathcal{H}_2,\acute{\circ})$ as following:

$\acute{\mathfrak{I}} = \{\phi ,\mathcal{H}_2,\{u\} \}$

We know that $\mathcal{P}^*(\mathcal{H}_2) = \{\{u\},\{v\},\{w\},\{u,v\},\{u,w\} ,\{v,w\},\mathcal{H}_2 \}$

Now, Define base for a topology on $\mathcal{P}^*(\mathcal{H}_2)$

$$\mathcal{S}_{\emptyset} = \{\emptyset\}$$
$$\mathcal{S}_{\{\mathcal{H}_2\}} = \mathcal{P}^*(\mathcal{H}_2)$$
$$\mathcal{S}_{\{u\}} = \{u\}$$
$$\Longrightarrow \acute{\mathfrak{I}}^* = \{\{\emptyset\}, \mathcal{P}^*(\mathcal{H}_2), \{u\}\}.$$

Now, to find $\mathcal{H}_2 \times \mathcal{H}_2$ and $\acute{\mathfrak{I}}_{\mathcal{H}_2\times\mathcal{H}_2}$.

$$\mathcal{H}_2 \times \mathcal{H}_2 = \{(u,u),(u,v),(u,w),(v,u),(v,v),(v,w),(w,u),(w,v),(w,w)\}.$$

$$\mathfrak{I}_{2\ \mathcal{H}_2\times\mathcal{H}_2} = \{ \emptyset,\mathcal{H}_2 \times \mathcal{H}_2,\{(u,u)\},\{(u,u),(u,v),(u,w)\} ,\{(u,u),(v,u),(w,u)\},$$
$$\{(u,u),(u,v),(u,w),(v,u),(w,u)\}\}.$$

$$\acute{\circ}^{-1}(\emptyset) = \emptyset \in \acute{\mathfrak{I}}_{\mathcal{H}_2\times\mathcal{H}_2}.$$
$$\acute{\circ}^{-1}(\mathcal{P}^*(\mathcal{H}_2)) = \mathcal{H}_2 \times \mathcal{H}_2 \in \acute{\mathfrak{I}}_{\mathcal{H}_2\times\mathcal{H}_2}$$
$$\acute{\circ}^{-1}(\{u\}) = \{(u,u)\} \in \acute{\mathfrak{I}}_{\mathcal{H}_2\times\mathcal{H}_2}$$

Hince $\acute{\circ}$ is continuous.

By Definition 2.3, we define $/: \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{P}^*(\mathcal{H}_2)$ as follows:

$$/^{-1}(\emptyset) = \emptyset \in \acute{\mathfrak{I}}_{\mathcal{H}_2\times\mathcal{H}_2}.$$

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$$/^{-1}\,(\mathcal{P}^*(\mathcal{H}_2)) = \mathcal{H}_2 \times \mathcal{H}_2 \in \acute{\mathfrak{J}}_{\mathcal{H}_2\times\mathcal{H}_2}$$
$$/^{-1}\,(\{u\}) = \{(u,u)\} \in \acute{\mathfrak{J}}_{\mathcal{H}_2\times\mathcal{H}_2}$$

Hince / is continuous. Therefore $(\mathcal{H}_2, \acute{\circ}, \acute{\mathfrak{J}})$ is a topological hypergroup.

Definition 3.4.[6]

Let $(\mathcal{H}_1,\circ), (\mathcal{H}_2,\acute{\circ})$ be two hypergroups and define the topologies $\mathfrak{J}, \acute{\mathfrak{J}}$ on $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. A mapping f from $\mathcal{H}_1$ to $\mathcal{H}_2$ is said to be a *good topological homomorphism* if for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{H}_1$

1- $f(\mathfrak{a} \circ \mathfrak{b}) = f(\mathfrak{a}) \acute{\circ} f(\mathfrak{b})$.
2- f is open.

A good topological homomorphism is an *isomorphism* if f is one-to-one and onto, and we say that $\mathcal{H}_1$ is topologically isomorphic to $\mathcal{H}_2$.

Recall that f is a homeomorphism if f is one-to-one - onto and both f and f^{-1} are continuous.

Example 3.5.

From Example 3.2. and Example 3.3. and using Definition 3.4. Define $f: (\mathcal{H}_1, \circ, \mathfrak{J}) \longrightarrow (\mathcal{H}_2, \acute{\circ}, \acute{\mathfrak{J}})$ by

$$f(e) = u \quad, f(a) = v \quad, f(b) = w$$

For instance, for $e, a, b \in \mathcal{H}_1$ we have

$$f(e \circ b) = f(b) = \{w\}.$$
$$f(e) \,\acute{\circ}\, f(b) = u \,\acute{\circ}\, w = \{w\}.$$

Then, f is an isomorphism, but f is not a homeomorphism because it is not continuous

$$f^{-1}(\{w\}) = \{b\} \notin \mathfrak{J} \quad .\square$$

Conclusion

Example 3.5. shows that an algebraic isomorphism between two topological hypergroups does not necessarily induce a homeomorphism. Therefore, algebraic and topological structures are independent in the general theory of topological hypergroups. Additional conditions are required in order to establish an equivalence between algebraic isomorphisms and homeomorphisms.

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