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Original article

The Direct Product of Hypermodules over a Hyperring

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Abstract
The purpose of this work is studying the direct product of hypermodules. We introduce a general definition of the direct product of R-hypermodules by extending the classical coordinatewise construction from module theory. It is shown that the direct product preserves the axioms of an R-hypermodule. This result helped us to prove that, direct Product of R-hypermodules is an R-hypermodule, and that the canonical projections as hyperhomomorphisms.
Keywords. *Hyperstructure, Hyperring, R-hypermodule, Direct Product.*

Introduction

Anvariyeh (2010) [1] investigated the case of 1-hypermodules and proved that their coordinatwise direct product is again a 1-hypermodule. The restriction to 1-hypermodules was mainly motivated by the relative simplicity of their structure: the hyperoperations are essentially single-valued or constrained, which allows one to avoid many of the technical complications that arise in the general setting of R-hypermodules. At that time, the categorical framework for hypermodules was not yet fully developed, so focusing on this special class provided a tractable context to establish the first results on direct products. In contrast, the present work extends this line of research by formulating a general definition of the direct product for arbitrary R-hypermodules and proving that it satisfies both the axioms of an R-hypermodule and the universal property, thereby placing it within the categorical setting of hypermodule theory. We shall recall some notions and basic results about hyperstructures [2] that we shall use in the following paragraphs.

Let H be a nonempty set and let $P^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation on* H is a map$\circ$: H ×H →P^* (H) and the couple (H, ∘) is called a *hypergroupoid*. If A and B are nonempty subsets of H, then we denote $A \circ B = \bigcup_{a\in A,\, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

A hyper groupoid (H, ∘) is said to be *commutative* if a ∘b = b ∘a for all a, b ∈ H. A hypergroupoid (H, ∘) is called a *semihypergroup* if for all x, y, z of H we have (x ∘ y) ∘ z = x ∘ (y ∘ z), which means that

$$u \circ z = \bigcup_{u\in x \circ y} u \circ z = \bigcup_{v\in y \circ z} x \circ v .$$

An element e of H is called an *identity* (scalar-identity) of (H,∘) if for all a ∈ H, we have a ∈ (e ∘a) ∩ (a ∘ e), ({a} = (e ∘ a) ∩ (a ∘ e)). We say that a semihypergroup (H, ∘) is a *hypergroup* if for all x ∈ H, we have x ∘ H = H ∘ x = H. A sub-hypergroup (K, ∘) of (H, ∘) is a nonempty set K, such that for all k ∈K, we have k ∘ K = K ∘ k = K. Several kinds of hyperrings and hypermodules can be defined on a nonempty set. In what follows, we shall consider some of the most general types of hyperrings and hypermodules.

Preliminaries

Definition 2.1.[3]. The triple (R, +, ·) is a *hyperring* if:
i) (R, +) is a commutative hypergroup;
ii) (R, ·) is a semi hypergroup;
iii) the hyperoperation "•" is distributive over the hyperoperation "+", which means that for all r, s, t of R we have:

$$r(s + t) = rs + rt \text{ and } (r + s)t = rt + st.$$

Example2.2.[6]. Let a ∈ R with a ≥ 1 and R = [a, ∞) ∪{0}. Define a hyperoperation ⊕ on

$$x \oplus y = \begin{cases} \{y\} & if \quad x = 0 \\ \{x\} & if \quad y = 0 \\ [a,\infty)\cup\{0\} & if \quad x=y\neq 0 \\ \{\min\{x,y\}\} & if \quad x\neq y,\ x\neq 0,\ y\neq 0 \end{cases}$$

Moreover, define a hyperoperation ⊙ on R by a ⊙ b = {a ·b}, where ·is the usual multiplication on R, for all a, b ∈ R. Then (R, ⊕, ⊙) is a strongly distributive hyperring.

Example2.3. Let R = {0, 1, 2} be a set with hyperoperation + and binary operation • as following:

+	0	1	2
0	0	1	2
1	1	1	R
2	2	R	2

•	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

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Then (R,+, •) is a hyperring.

Definition 2.4.[3] Let (R, *, ∘) be a hyperring. A nonempty set M, endowed with two hyper operations ⊕⊙, is called a *left hypermodule* over (R,*, ∘) if the following conditions hold:

i) (M,⊕) is a commutative hyper group;

ii) ⊙: R×M →P*(M) acts such that for all a, b of M and r, s of R we have

1) r ⊙(a ⊕ b) = (r ⊙ a) ⊕ (r ⊙ b)
2) (r * s) ⊙a = (r ⊙ a) ⊕ (s ⊙ a)
3) (r ∘ s) ⊙ a = r ⊙ (s ⊙ a).

If both (R, *) and (M , ⊕) have scalar identities, denoted by 0_R and 0_M , then the R-hypermodule (M , ⊕, ⊙) also satisfies the condition: for all a of M, we have 0_R ⊙ a = 0_M , Moreover, if (R, ∘) has an identity, denoted by 1, then the hyper module (M , ⊕, ⊙) is called unitary if it satisfies the condition: for all a of M, we have ⊙a = a. In what follows, we consider only left hypermodules, which we shall simply call *R-hypermodules*.In (i, ii), if the equalities r ⊙ (a ⊕ b) ⊆ (r ⊙ a) ⊕ (r ⊙ b) and (r * s) ⊙ a ⊆ (r ⊙ a) ⊕ (s ⊙ a) hold, then the R-hypermodule M is said to be *strongly distributive*.

Example 2.5. In Example 2.3. Let M_1 = {a, b} be a set with hyperoperation ⊕ as follows:

⊕	a	b
a	a	b
b	b	M_1

⊙	a	b
0	a	a
1	a	b
2	a	b

Then (M_1, ⊕) is a commutative hypergroup. Now, we define the acts $\odot: R \times M_1 \rightarrow$ P*(M) as the previous table, then (M_1, ⊕, ⊙) is an R-hypermodule.

Similarly**,** M_2={c, d}, and M_3={f, g} with hyperoperation acts as follows:

⊕	c	d
c	c	d
d	d	M_2

⊙	c	d
0	c	c
1	c	d
2	c	d

⊕	f	g
f	f	g
g	g	M_3

⊙	f	g
0	f	f
1	f	g
2	f	g

Then (M_2, ⊕, ⊙) and (M_3, ⊕, ⊙) are R-hypermodules.

Definition 2.6.[1][3] A nonempty subset N of M is called a *sub-hyper-module* of the hypermodule (M,+,·) if (N,+) is a sub-hypergroup of (M, +) and RN ⊆ P*(N).

Example 2.7.[4] Let R be the hyperring defined in Example 2.2. Let t ∈ R with 0 < t ≤1 and M = [0, t). Define a hyperoperation + on M by, for any x, y ∈ M

$$x + y = \begin{cases} \{\max\{x, y\}\} & if \quad x \neq y \\ [0, x] & if \quad x = y \end{cases}$$

In addition, define a multivalued scalar operation ⊙ by, for any a ∈ R and x ∈ M

$$x \odot y = \begin{cases} \{0\} & if \quad a = 0 \\ \left[0, \frac{x}{a}\right] & if \quad a \neq 0 \end{cases}$$

It can be checked directly that (M, +, ⊙) is a strongly distributive R-hypermodule. Moreover, {0} is a sub-hypermodule of M.

Definition 2.8.[1] Let M_1 and M_2 be R-hypermodules. A function f: M_1 →M_2 is called an *R-homomorphism,* if for every (x, y) ∈M and r ∈ R we have

1) f (x + y) = f (x) + f (y) and
2) f (r ·x) = r ·f (x).

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Results

Let $\prod_{i\in I} M_i$ be the direct product of the commutative hypergroups M_i, where $\prod_{i\in I} M_i$ ={ α |α: I →$\bigcup_{i\in I} M_i$ via α(i) ∈ M_i for all i ∈ I}. For every i ∈I, let us denote α(i) := m_i and α= $(m_i)_{i\in I}$. Here, (i ∈ I) is called the i^{th} component of α. The function α= $(m_i)_{i\in I}$.= (m_1, m_2 ,..., m_i ,...) in case I is a countable set.
Let α = $(m_i)_{i\in I}$. , β = $(m'_i)_{i\in I}$. ∈ $\prod_{i\in I} M_i$. Functioning by the definition of equality, α = β if and only if m_i = m'_i` for every i ∈I.

Remarks 3.1.[5][6]
(i) If we are given maps $α_i$: A → A_i , i ∈ I, then the natural map α : A → $\prod_{i\in I} A_i$ is defined by αa(i) = $α_i$ a
(ii) If we are given maps $α_i$: A_i →B_i, then the natural map α: A$\prod_{i\in I} A_i$ → B$\prod_{i\in I} B_i$, is defined by αa(i) = $α_i$ (a(i))

Definition 3.2. Let R be a hyperring, and let be a family of R-hypermodules. We defined the *direct product* of this family as the set follows:

$$P = \prod_{i\in I} M_i := \{(m_i)_{i\in I} \mid m_i \in M_i \ \forall \ i \in I\}$$

On this set, we define the following coordinate-wise operations:
Hyperaddition: for m = (m_i), n = (n_i) ∈ $\prod M_i$,

$$m \mathbin{\widehat{\oplus}} n := (m_i \oplus_i n_i)=\{(x_i)_{i\in I}\in \prod_{i\in I} M_i \mid x_i \in m_i \oplus_i n_i \text{for all } i \in I \} \qquad (3.1)$$

Scalar action: For r ∈ R and m = (m_i) ∈ $\prod M_i$,,

$$r \mathbin{\widehat{\odot}} m:=(r \odot_i m_i) =\{(y_i)_{i\in I}\in \prod_{i\in I} M_i \mid y_i \in r \odot_i m_i \quad \text{for all } i \in I \} \qquad (3.2)$$

Identity element: 0_P:= $(0_i)_{i\in I}$; 0_i ∈ M_i With these operations, we can prove in the next theorem (P, $\widehat{\oplus}$, $\widehat{\odot}$) is an R-hypermodule.

Proposition 3.3. The hyper operation $\widehat{\oplus}$: $\prod M_i \times \prod M_i \to P^*(\prod M_i)$ as
m $\widehat{\oplus}$ n := ($m_i \oplus_i n_i$) = {$(x_i)_{i\in I}\in \prod_{i\in I} M_i$ | $x_i\in m_i \oplus_i n_i$ for all i ∈ I }, is well-define.

Prove. Let. m = m` and n = n`
Then $m_i \oplus_i n_i$ = m_i` $\oplus_i$ $n_i$ `
Hence m $\widehat{\oplus}$ n = m` $\widehat{\oplus}$ n`. Therefore $\widehat{\oplus}$ is well-defined.

Theorem 3.4. The (P, $\widehat{\oplus}$, $\widehat{\odot}$) is a left R-hypermodule

Proof. *First, we must prove (P,* $\widehat{\oplus}$ *) is a commutative hypergroup*
for all m, n ∈ P , i ∈ I we have $m_i \oplus_i n_i \neq \emptyset$, by the axiom of M_i.

$$\Rightarrow m \mathbin{\widehat{\oplus}} n \neq \emptyset$$

we have $\oplus_i$ is commutative over M_i, since $m_i \oplus_i n_i$ = $m_i \oplus_i n_i$
then m $\widehat{\oplus}$ n = $m_i \oplus_i n_i$ = $m_i \oplus_i n_i$ = n $\widehat{\oplus}$ m
for all s, m, n ∈ P, s $\widehat{\oplus}$ (m $\widehat{\oplus}$ n) = (s $\oplus_i$ ($m_i \oplus_i n_i$)), since $\oplus_i$ is associative

$$\Rightarrow (s \oplus_i m_i) \oplus_i n_i$$
$$= (s \mathbin{\widehat{\oplus}} m) \mathbin{\widehat{\oplus}} n$$

For all i ∈I, there exists $0_i \in M_i$, 0_P = (0_i) then m $\widehat{\oplus}$ 0_P = ($m_i \oplus_i 0_i$) = (m_i) = m
For all I there exists $\tilde{m}$ such that 0_i∈ ($m_i \oplus_i \tilde{m}_i$) defined = $(\tilde{m}_i)_i$ then 0_P ∈ (m $\widehat{\oplus}$ $\tilde{m}$)
(We call the element $\tilde{m}$ the opposite of m)
By satisfying the properties of a commutative hypergroup, then (P, $\widehat{\oplus}$ *) is a commutative hypergroup*
Second, we must prove that (P, $\widehat{\oplus}$ *,* $\widehat{\odot}$ *) is an R-hypermodule***.**
r $\widehat{\odot}$ (m $\widehat{\oplus}$ n) = r $\odot_i$ ($m_i \oplus_i n_i$), since $\odot_i$ is distributive over $\oplus_i$ then

$$(r \odot_i m_i) \oplus_i (r \odot_i n) = (r \mathbin{\widehat{\odot}} m)(r \mathbin{\widehat{\odot}} n)$$

(r $\widehat{\oplus}$ s) $\widehat{\odot}$ m = (r $\oplus_i$ s) $\odot_i$ m for all r, s ∈R m ∈M by axiom of M, we have

$$(r \odot_i m_i) \oplus_i (s \odot_i m_i) = (r \mathbin{\widehat{\odot}} m) \mathbin{\widehat{\oplus}} (s \mathbin{\widehat{\odot}} m)$$

(rs) $\widehat{\odot}$ m = (rs) $\odot_i$ m_i⊙ m by axiom of M_i we have (r $\odot_i$ (s $\odot_i$ m)) = (r $\widehat{\odot}$ (s $\widehat{\odot}$ m))
since 1 $\odot_i$ m_i = {(m_i)} for all 1 ∈ R then 1 $\widehat{\odot}$ m = m
then (P, $\widehat{\oplus}$, $\widehat{\odot}$) **are** R-hypermodule.□

Example 3.5. In Examples 2.3 and 2.5. , let
$P = \prod_{i=1}^{3} M_i = M_1 \times M_2 \times M_3$={$(m_1, m_2, m_3)$ ∀ $m_i \in M_i, i = 1,2,3$}
= {a, b}× {c, d} ×{f, g}={ (a,c,f), (a,c,g),(a,d,f), (a,d,g), (b,c,f), (b,c,g), (b,d,f), (b,d,g)}. By (3.1) the hyperoperation $\widehat{\oplus}$ on P is defined as follows:

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$\widehat{\oplus}$	(a,c,f)	(a,c,g)	(a,d,f)	(a,d,g)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)
(a,c,f)	(a,c,f)	(a,c,g)	(a,d,f)	(a,d,g)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)
(a,c,g)	(a,c,g)	{(a,c,f), (a,c,g)}	(a,d,g)	{(a,d,f), (a,d,g)}	(b,c,g)	{(b,c,f), (b,c,g)}	(b,d,g)	{(b,d,f), (b,d,g)}
(a,d,f)	(a,d,f)	(a,d,g)	{(a,c,f), (a,d,f)}	{(a,c,g), (a,d,g)}	(b,d,f)	(b,d,g)	{(b,c,f), (b,d,f)}	{(b,c,g), (b,d,g)}
(a,d,g)	(a,d,g)	{(a,d,f), (a,d,g)}	{(a,c,g), (a,d,g)}	{(a,c,f), (a,c,g), (a,d,f), (a,d,g)}	(b,d,g)	{(b,d,f), (b,d,g)}	{(b,c,g), (b,d,g)}	{(b,c,f), (b,c,g), (b,d,f), (b,d,g)}
(b,c,f)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)	{(a,c,f), (b,c,f)}	{(a,c,g), (b,c,g)}	{(a,d,f), (b,d,f)}	{(a,d,g), (b,d,g)}
(b,c,g)	(b,c,g)	{(b,c,f), (b,c,g)}	(b,d,g)	{(b,d,f), (b,d,g)}	{(a,c,g), (b,c,g)}	{(a,c,f), (a,c,g), (b,c,f), (b,c,g)}	{(a,d,g), (b,d,g)}	{(a,d,f), (a,d,g), (b,d,f), (b,d,g)}
(b,d,f)	(b,d,f)	(b,d,g)	{(b,c,f), (b,d,f)}	{(b,c,g), (b,d,g)}	{(a,d,f), (b,d,f)}	{(a,d,g), (b,d,g)}	{(a,c,f), (a,d,f), (b,c,f), (b,d,f)}	{(a,c,g) (a,d,g), (b,c,g), (b,d,g)}
(b,d,g)	(b,d,g)	{(b,d,f), (b,d,g)}	{(b,c,g), (b,d,g)}	{(a,c,f), (b,c,g), (b,d,f), (b,d,g)}	{(a,d,g), (b,d,g)}	{(a,d,f), (a,d,g), (b,d,f), (b,d,g)}	{(a,c,g), (a,d,g), (b,c,g), (b,d,g)}	P

Then (P, $\widehat{\oplus}$) is a commutative hypergroup.
Now. By (3,2) an acts $\widehat{\odot}: R \times P \rightarrow$ P*(P) is defined by the following table:

$\widehat{\odot}$	(a,c,f)	(a,c,g)	(a,d,f)	(a,d,g)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)
0	(a,c,f)	(a,c,f)	(a,c,f)	(a,c,f)	(a,c,f)	(a,c,f)	(a,c,f)	(a,c,f)
1	(a,c,f)	(a,c,g)	(a,d,f)	(a,d,g)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)
2	(a,c,f)	(a,c,g)	(a,d,f)	(a,d,g)	(b,c,f)	(b,c,g)	(b,d,f)	(b,d,g)

Then (P, $\widehat{\oplus}$, $\widehat{\odot}$) is an R-hypermodule
.

Proposition 3.6. The map π_j : P →M_j by $\pi_j(m_i)_{i\in I}$ = m_j for j∈I, is a natural projection homomorphism.

Proof.

if x ∈ m $\widehat{\oplus}$ n ⇒ $x_i \in m_i \oplus_i n_i$ then π(m $\widehat{\oplus}$ n) = π($m_i \oplus_i n_i$) = π(m_i) $\oplus_i$ π(n_i)

= π(m) $\widehat{\oplus}$ π(n)

if y ∈ r $\widehat{\odot}$ m ⇒ $y_i \in$ r $\odot_i$ m_i then π(r $\widehat{\odot}$ m) = π(r $\odot_i$ m_i)

= r $\odot_i$ π(m_i).□

Conclusion

In this paper, we proved that, direct Product of R-hypermodules is an R-hypermodule, and that the canonical projections are hyperhomomorphisms.

References

1. Anvariyeh SM, Davvaz B. On the heart hypermodules. Math Scand. 2010;106:161-8.
2. Davvaz B, Vougiouklis T. A walk through weak hyper-structures: Hv-structures. Singapore: World Scientific Publishing Company; 2019.
3. Leoreanu-Fotea V. Fuzzy hypermodules. Comput Math Appl. 2009;57(3):466-75.
4. Wongyai C, Pianskool S. Some properties of extending hypermodules and C11-hypermodules. J Math Res. 2023;15(3):13-26. doi:10.5539/jmr.v15n3p13.
5. Türkmen E, Türkmen BN, Bordbar H. A characterization of normal injective and normal projective hypermodules. Axioms. 2024;13(5):410. doi:10.3390/axioms13050410.
6. Burris S, Sankappanavar HP. A course in universal algebra. Millennium ed. New York: Springer; 2012

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