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Original article

Maximum Matching of Zero-Divisor Graph Over a Commutative Ring

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Abstract
The zero-divisor graph $\Gamma(Z_n)$ is constructed by taking the nonzero zero-divisor of a commutative ring Z_n as vertices, with the edges connecting two vertices if their product is zero in Z_n. In this paper, we investigate $\alpha'(\Gamma(Z_n))$ and $\alpha(\Gamma(Z_n))$, the matching number and independence number of the zero-divisor graph $\Gamma(Z_n)$, respectively, for several values of n, when $n = kq$, with integer $k \in \{2,3,5,7\}$, and when $n = pq$, where p and q are distinct prime numbers and $p < q$. We prove that in these cases, the graph $\Gamma(Z_n)$ is isomorphic to complete bipartite graphs, which allows for the exact determination of $\alpha'(\Gamma(Z_n))$ and $\alpha(\Gamma(Z_n))$. This study demonstrates the relationship between the algebraic structure of Z_n and the graph-theoretic properties of its zero-divisor graph.
Keywords. Maximum Matching, Zero-Divisor, Commutative Ring.

Introduction

The intersection of algebra and graph theory has created a useful area of study where algebraic structures can be understood using graph-based methods. One of these studies was introduced by I. Beck in 1988 [1]. Let $Z_n$denote a commutative ring. Define $\Gamma(Z_n)$ to be a graph with vertex set $V(\Gamma(Z_n) = Z_n$ and the vertices u and v of $\Gamma(Z_n\ )$ are adjacent if $uv \equiv 0 \ mod\ (n)$, otherwise u, v are non-adjacent. This study and other studies motivated the work of D. Anderson and P. Livingston in [2], who investigated the connectivity of a zero-divisor graph with a small diameter and characterized the rings whose zero-divisor graph is a complete or a star. In 2012, R. Sankar, J, and Meena [3] evaluated the connected domination number of Z_n in some cases of n. They characterized the domination number $\gamma(\Gamma(Z_n))$ of graphs to find out $\gamma(\Gamma(Z_n)) = \gamma_c(\Gamma(Z_n))$, and they found out that m is the connected domination number of $\Gamma(Z_{p_1^{e_1} \times p_2^{e_2} \times \ldots \times p_m^{e_m}})$. In [4], S. Suthar and O. Prakash studied matchings in $\Gamma(Z_n\ )$ and in the line graph $L(\Gamma(Z_n))$ of $\Gamma(Z_n)$. In the same paper, they studied the relationship between the perfect graph and perfect matching. The previous studies motivate us to prove the following main theorem of this paper, which evaluates the stability number $\alpha(\Gamma(Z_n))$ and the matching number $\alpha'(\Gamma(Z_n))$ of $\Gamma(Z_n)$.

Theorem 1.1. Let $\Gamma(Z_n)$ be a zero-divisor graph and p, q are prime numbers with $p < q$. If $n = pq$, then $\alpha(\Gamma(Z_n)) = q - 1$ and $\alpha'(\Gamma(Z_n)) = p - 1$.

Throughout this paper, we follow [5] for undefined terms for graphs. A graph G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a nonempty set of vertices of G and $E(G)$ is a set of edges. A link is an edge e in a graph G incident with vertices u and u, define $V(e) = \{u, v\}$. For a nonempty subset of vertices $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. A subset S of $V(G)$ is called a stable set of G if there are no edges between any pair of vertices of S. A stable set in a graph G is maximum if the graph contains no larger stable set. The cardinality of a maximum stable set in a graph G is called the stability number of G, denoted $\alpha(G)$. A matching in a graph G is a set of pairwise nonadjacent links. A maximum matching of G is the matching that covers as many vertices as possible. The number of edges in a maximum matching of a graph G is called the matching number of G and denoted $\alpha'(G)$. For a graph G, let M be a matching in a graph G. An M −alternating path in G is a path whose edges are alternately in M and $E \backslash M$. An M -alternating path might or might not start or end with edges of M. If neither its origin nor its terminus is covered by M the path is called an M −augmenting path.

The following theorem is the fundamental theorem, known as Berge's theorem, which shows an important connection between maximum matchings and M −augmenting paths.

Theorem 1.2. (Berge [6]) A matching M of a graph G is a maximum matching if and only if G does not have M −augmenting paths.

Proof of Theorem 1.1.

Lemma 2.1. Let G be a complete bipartite graph with partition sets S and T where $|S| = n$ and $|T| = m$ and $n \neq m$. Then $\alpha(G) = \{n, m\}$ and $\alpha'(G) = \{n, m\}$.

Proof: Assume $n < m$. As S and T are independent sets, obviously that $\alpha(G) = m$. Let M be a maximum matching of G. As G is a bipartite graph, then for any $e \in M$, $|V(e) \cap S| = 1$ and $|V(e) \cap T| = 1$, so all the vertices of S must be covered by M. It follows $n \leq |M|$. By a contradiction, we assume that M be a maximum matching of G with $|M| = n + 1$ where $n + 1 \leq m$. Assume $M = \{e_i = u_i v_i\colon u_i \in S, v_i \in T \text{ for } i \in \{1, 2, \ldots, n + 1\}\}$. Consider the edge $e_{n+1} = u_{n+1} v_{n+1} \in E(G) \cap M$. By the definition of G and the size of S, then the vertex $u_{n+1} = u_k$ for some $k \in \{1, 2, \ldots, n\}$, let $u_{n+1} = u_1$. As $u_1 v_n, u_n v_{n+1} \in E(G) - M$ and $e_{n+1} = u_1 v_{n+1} \in M$, then set of edges $\{u_1 v_n, u_1 v_{n+1}, u_n v_{n+1}\}$ induces an M-augmenting path. Since M is a maximum matching, by Theorem 1.2, G

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does not have an M-augmenting path, thus the edge $e_{n+1} \notin M$. Therefore $|M| = n = \alpha'(G)$. ∎

Throughout this paper, we assume that all the zero divisor graphs over Z_n are simple and do not contain the vertex 0. For prime numbers p and q.

Theorem 2.2. Let $\Gamma(Z_n)$ be a zero-divisor graph. If $n = 2q$ where $q > 2$, then $\alpha\left(\Gamma(Z_{2q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{2q}\right)\right) = 1$.

Proof: Let $V = \{q, 2, 4, 6, \ldots, 2(q-1)\}$ be the vertex set of a zero-divisor graph $\Gamma(Z_{2q})$. Let $u = q$ and for any $v \in \{2, 4, \ldots, 2(q-1)\}$, $uv \equiv 0 \ mod\ (2q)$, thus $uv \in E\left(\Gamma\left(Z_{2q}\right)\right)$. For any $v, v' \in \{2, 4, \ldots, 2(q-1)\}$, $vv' \not\equiv 0\ mod\ (2q)$, so $vv' \notin E\left(\Gamma\left(Z_{2q}\right)\right)$, it implies that $\{2, 4, \ldots, 2(q-1)\}$ is a stable set. Therefore, $V_1 = \{q\}$ and $V_2 = \{2, 4, \ldots, 2(q-1)\}$ are the partition sets of $V\left(\Gamma\left(Z_{2q}\right)\right)$, thus $\Gamma(Z_{2q})$ is an isomorphic to the complete bipartite graph $K_{1,q-1}$. As $|V_1| = 1$ and $|V_2| = q - 1 > 1$, by Lemma 2.1, $\alpha\left(\Gamma(Z_{2q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{2q}\right)\right) = 1$. Which completes the proof of Theorem 2.2.∎

Theorem 2.3 Let $\Gamma(Z_n)$ be a zero-divisor graph. If $n = 3q$ where $q > 3$, then $\alpha\left(\Gamma(Z_{3q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{3q}\right)\right) = 2$.

Proof: Let $V = \{q, 2q, 3, 6, 9, \ldots, 3(q-1)\}$ be the vertex set of zero divisor graph $\Gamma(Z_{3q})$. By the definition of zero-divisor graph $\Gamma(Z_{3q})$, $uv \equiv 0\ mod\ (3q)$, for any $u \in \{q, 2q\}$ and for any $v \in \{3, 6, 9, \ldots, 3(q-1)\}$, so $uv \in E\left(\Gamma\left(Z_{3q}\right)\right)$. As $q.2q = 2q^2 \not\equiv 0\ mod\ (3q)$, then the set of vertices $V_1 = \{q, 2q\}$ is a stable set of $\Gamma(Z_{3q})$ of size 2. Likewise, for any $v, v' \in \{3, 6, 9, \ldots, 3(q-1)\}$, $vv' \not\equiv 0\ mod\ (3q)$, so the set of vertices $V_2 = \{3, 6, 9, \ldots, 3(q-1)\}$ is a stable set of $\Gamma(Z_{3q})$ of size $q - 1 > 3$, where $V_1 \cup V_2 = V\left(\Gamma\left(Z_{3q}\right)\right)$. So, V_1, V_2 are the partition sets of $V\left(\Gamma\left(Z_{3q}\right)\right)$, thus $\Gamma(Z_{3q})$ is an isomorphic to the complete bipartite graph $K_{2,q-1}$. By Lemma 2.1, $\alpha\left(\Gamma(Z_{3q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{3p}\right)\right) = 2$.∎

Theorem 2.4. Let $\Gamma(Z_n)$ be a zero-divisor graph. If $n = 5q$ where $q > 5$, then $\alpha\left(\Gamma(Z_{5q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{5q}\right)\right) = 4$.

Proof: Let $V = \{q, 2q, 3q, 4q, 5, 10, \ldots, 5(q-1)\}$ be the vertex set of zero divisor graph $\Gamma(Z_{5q})$. By the definition of zero-divisor graph $\Gamma(Z_{5q})$, $uv \equiv 0\ mod\ (5q)$, for any $u \in \{q, 2q, 3q, 4q\}$ and for any $v \in \{5, 10, \ldots, 5(q-1)\}$, so $uv \in E\left(\Gamma\left(Z_{5q}\right)\right)$. And for any vertices $u, u' \in V_1 = \{q, 2q, 3q, 4q\}$, $uu' \not\equiv 0\ mod\ (5q)$, So the vertex set V_1 is a stable set of size 4. Likewise, for any $v, v' \in V_2 = \{5, 10, \ldots, 5(q-1)\}$, $vv' \not\equiv 0\ mod\ (3q)$, so V_2 is a stable set of size $q - 1$. As V_1, V_2 are the partition sets of $V\left(\Gamma\left(Z_{5q}\right)\right)$, then $\Gamma(Z_{5q})$ is isomorphic to the complete bipartite graph $K_{4,q-1}$. By Lemma 2.1, $\alpha\left(\Gamma(Z_{5q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{5q}\right)\right) = 4$.∎

Theorem 2.5. Let $\Gamma(Z_n)$ be a zero-divisor graph. If $n = 7q$ where $q > 7$, then $\alpha\left(\Gamma(Z_{7q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{7q}\right)\right) = 6$.

Proof:

Let $V = \{q, 2q, 3q, 4q, 5q, 6q, 7, 14, \ldots, 7(q-1)\}$ be the vertex set of the zero-divisor graph $\Gamma(Z_{7q})$. By the definition of a zero-divisor graph $\Gamma(Z_{7q})$, $uv \equiv 0\ mod\ (7q)$ for any $u \in \{q, 2q, 3q, 4q, 5q, 6q\}$ and for any $v \in \{7, 14, \ldots, 7(q-1)\}$, so $uv \in E\left(\Gamma\left(Z_{7q}\right)\right)$. Otherwise, for any vertices $u, u' \in V_1 = \{q, 2q, 3q, 4q, 5q, 6q\}$, $uu' \not\equiv 0\ mod\ (7q)$, which implies $uu' \notin E\left(\Gamma\left(Z_{7p}\right)\right)$, so the vertex set V_1 is a stable set of size 6. Likewise, for any $v, v' \in V_2 = \{7, 14, \ldots, 7(q-1)\}$, $vv' \not\equiv 0\ mod\ (7q)$, which implies $vv' \notin E\left(\Gamma\left(Z_{7q}\right)\right)$, so V_2 is a stable set of size $q - 1$. Since V_1, V_2 are the partition sets of $\Gamma(Z_{7q})$, thus $\Gamma(Z_{7q})$ is isomorphic to the complete bipartite graph $K_{6,q-1}$. By Lemma 2.1, $\alpha\left(\Gamma(Z_{7q})\right) = q - 1$ and $\alpha'\left(\Gamma\left(Z_{7q}\right)\right) = 6$. ∎

To prove Theorem 1.1, let $\Gamma(Z_n)$ be a zero-divisor graph, if $n = pq$ where p, q are prime numbers with $q > p > 2$. Assume $V = \{p, 2p, \ldots, p(q-1), q, 2q, \ldots, q(p-1)\}$, be the vertex set of the zero-divisor graph $\Gamma(Z_{pq})$. Note that for any $u \in V_1 = \{p, 2p, \ldots, p(q-1)\}$, and for any $v \in V_2 = \{q, 2q, \ldots, q(p-1)\}$, $uv \equiv 0\ mod\ (pq)$, so $uv \in E\left(\Gamma\left(Z_{pq}\right)\right)$. Thus, every vertex in V_1 is adjacent to every vertex in V_2. Moreover, for any two vertices $u, u' \in V_1$, $uu' \not\equiv 0\ mod\ (pq)$. Thus the vertex set V_1 is a stable set of size $q - 1$. Similarly, V_2 is a stable set of size $p - 1$. Therefore, the zero-divisor graph $\Gamma(Z_{pq})$ is isomorphic to the complete bipartite graph $K_{p-1,q-1}$. As $p < q$, by Lemma 2.1, $\alpha\left(\Gamma(Z_{pq})\right) = q - 1$ and $\alpha'\left(\Gamma(Z_n)\right) = p - 1$. This completes Theorem 1.1.∎

Conclusion

This study determined the matching number investigated and the stability number of the zero-divisor graph over for, with integer, and for where and are distinct prime numbers, and we proved that these graphs are isomorphic to a complete bipartite graph. The results show a strong connection between the algebraic structure of and the way its zero divisors behave, providing a useful basis for exploring other related graph properties in future studies.

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