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Original article

A Fractional α-Transform for Solving Second-Order Fractional Differential Equations: A New Insight

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Abstract
Fractional Differential Equation (FDE) plays a powerful role in the applications of applied science and engineering. In this paper, the α-fractional transform has been used to convert FDE to ordinary differential equations (ODEs). A new α-transform has been developed using the features of the special version of the α-fractional derivative for real functions. Studies demonstrated that this definition's elements are appropriate, relevant, and useful in solving FDEs and their applications. The proposed α-transform approach was used to convert the FDE to an ODE. Also, some of the examples of second-order FDEs are solved using this proposed method. This study presented the α-transform approach, which has been used for dealing with a number of FDE issues. Therefore, the examples show that the α-transform technique is the most cost-efficient and efficient method for converting FDEs into ODEs.
Keywords. RKN, DEs, ODEs, FDEs, A-Transform, A-Fractional Derivative.

Introduction

Classical partial and ordinary differential equations are generalized to fractional differential equations (FDEs), in which the derivatives may be of any real or complex order. The more sophisticated technique for modeling systems with memory and hereditary features is fractional calculus, a powerful area of mathematics. Due to fractional derivatives are intrinsically non-local, as opposed to local operators like integer-order derivatives, a system's future state is contingent upon its complete past. FDEs are particularly well-suited to explaining situations when conventional models are inadequate because of this special quality. One major use is in viscoelasticity, a feature that fractional-order derivatives are excellent at capturing. Materials such as polymers, gels, and biological tissues may behave both viscously and elastically. FDEs are essential in electrochemistry because they may be used to represent the intricate behavior of batteries and supercapacitors, especially when ion transport across porous electrodes is being described. Control theory also makes use of fractional-order controllers that are more flexible and durable than their integer-order equivalents when it comes to handling complicated systems.

FDEs are used in biology and biophysics to simulate anomalous diffusion in cells, which deviates from normal Brownian motion when particle transport is impeded by a crowded environment. Neuronal signaling and the electrical characteristics of heart tissue are also modeled using them. Beyond these, FDEs are used extensively in image processing for improved edge recognition and texture analysis, in finance for modeling options pricing with long-range memory effects, and in signal processing for creating sophisticated filters. In the end, fractional differential equations provide a more precise and potent mathematical foundation for comprehending the intricate, time-dependent dynamics seen in a wide range of engineering and scientific fields.

Fractional derivative definitions were introduced by several authors. Various authors have examined and presented the properties and techniques for solving FDEs, as well as the theory of FDEs and their applications [1–5]. Several authors have contributed definitions of fractional derivatives. A new definition of the α-fractional integral and derivative of real functions was introduced by Mechee et al. [6], while other definitions of the fractional derivative were presented by Khalil et al. [7] and Zheng et al. [8]. For more reviews on this field, some authors in [9-15] solved some types of FDEs using analytical or numerical methods. Additionally, Unal et al. solved the variable coefficients and homogeneous sequential linear conformable FDEs of order two using the power series around an established point, and Abdel Jawad [16] developed the definition of the fractional conformable derivative while establishing basic ideas in the fractional calculus. Likewise, they established conformable fractional Hermite DEs [17]. In addition, Ortega and Rosales [18] proposed fractional conformable derivative characteristics, while Qasim and Holel [19] examined the oversight of particular composition fractional order DEs types that were significant to optimal control problems. Lastly, Euler and Runge-Kutta approaches have been expanded by the authors in [20–22] to solve particular types of FDEs.

This work uses the α-fractional derivative qualities of the new definition of α-fractional integral and derivative of the functions by Mechee et al. [6] to construct a novel α-transform. Using the α-transfer transform, FDE may be transformed into an ODE. The definition in [6] is probably the most effective definition for converting the FDE into ODE, according to the proposed method.

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Definitions

Definition 1: General FDEs of nth-Order

The general FDE of n^{th} -order is given as follows:

$$\emptyset\big(\varsigma, \omega(\varsigma), \omega^{(k)}(\varsigma), \omega^{(k\alpha)}(\varsigma); k = 1,2 \dots, n\big) = 0; \;\; a \le \varsigma \le b, \tag{1}$$

with the ICs: $\omega^{(k\alpha)}(a) = \beta_i$, for $k = 0,1,2, \dots, n.$ (2)

Definition 2: Quasi-Linear FDEs of nth Order

The quasi-linear FDE of n^{th} -order is given as follows:

$$\omega^{(n\alpha)}(\varsigma) = \Phi\big(\varsigma, \omega(\varsigma), \omega^{(k)}(\varsigma), \omega^{(n)}(\varsigma), \qquad \omega^{(k\alpha)}(\varsigma); j = 1,2 \dots, n-1\big); \; a \le \varsigma \le b, \tag{3}$$

with the ICs in Equation (2).

Definition 3: Quasi-Linear FDEs of Second-Order

The quasi-linear FDE of second order is given as follows:

$$\omega^{(2\alpha)}(\varsigma) = \Phi\left(\varsigma, \omega(\varsigma), \omega'(\varsigma), \omega''(\varsigma), \omega^{(\alpha)}(\varsigma)\right); \qquad a \le \varsigma \le b, \tag{4}$$

with the ICs: $\omega^{(0)}(a) = \beta_0$, $\omega^{(\alpha)}(a) = \beta_{1.}$ (5)

Fractional Derivatives

The α-fractional-derivative of the function φ (ς) and its properties, which have been investigated by [6], are presented in the following, as well as additional definitions of fractional derivatives.

Definition 4: α-Fractional-Derivative [6]

For the function $\varphi(\varsigma): [a, \infty) \to \mathcal{R}$ Mechee et al. [6] define the α-fractional-derivative as follows:

$$T_\alpha\big(\varphi(\varsigma)\big) = \varphi^{(\alpha)}(\varsigma) = \lim_{\epsilon\to 0} \frac{\varphi(\varsigma + \epsilon\varsigma^{\,1-\alpha}) - \varphi(\varsigma - \epsilon\varsigma^{\,1-\alpha})}{2\epsilon}. \tag{6}$$

for $\alpha \in (0,1]$.

Definition 5: Conformable Fractional-Derivative [7]

According to Khalil et al. [7], the attractive fractional derivative of the function $\varphi(\tau): [a, \infty) \to \mathcal{R}$. is defined as follows:

$$T_\alpha\big(\varphi(\varsigma)\big) = \varphi^{(\alpha)}(\varsigma) = \lim_{\epsilon\to 0} \frac{\varphi(\varsigma + \epsilon\varsigma^{\,1-\alpha}) - \varphi(\varsigma)}{2\epsilon}. \tag{7}$$

for $\alpha \in (0,1]$.

The Properties of α-Fractional-Derivative

The properties of the fractional derivative of the functions $\phi(\tau)$ and $\varphi(\tau)$ satisfy the following properties,

1. $T_1(\gamma\,\phi + \beta\,\varphi)(\varsigma) = \gamma\, T_1\big(\phi(\varsigma)\big) + \beta\, T_1\big(\varphi(\varsigma)\big),\ \gamma, \beta \in \mathcal{R}$
2. $T_1(\varsigma^q) = q\varsigma^{q-1}, q \in \mathcal{R},$
3. $T_1(\phi\,\varphi)(\varsigma) = \phi(\varsigma)\, T_1(\varphi(\varsigma)) + \varphi(\varsigma)\, T_1(\phi(\varsigma)),$
4. $T_1\left(\frac{\varphi}{\phi}(\varsigma)\right) = \frac{\phi(\varsigma)\, T_1(\varphi(\varsigma)) - \varphi(\varsigma) T_1(\phi(\varsigma))}{\phi^2(\varsigma)},$
5. $T_1(\varphi(\varsigma)) = 0$ Regarding to constant function $f(\varsigma) = \sigma,\ \sigma \in \mathcal{R}$. For $\alpha = 1$

Theorem 2.2 [6]

Consider $\varphi(\varsigma)$, and ϕ (ς) are α-fractional-differentiable functions at the point? ς . Then,

1. $T_\alpha(\varsigma^q) = q\,\varsigma^{q-1},\ q \in \mathcal{R},$
2. $T_\alpha(\varphi\,\phi)(\varsigma) = \varphi(\varsigma)\, T_\alpha\big(\phi(\varsigma)\big) + \phi(\varsigma)\, T_\alpha(\varphi(\varsigma)),$
3. $T_\alpha\left(\frac{\varphi}{g}(\varsigma)\right) = \frac{\phi(\varsigma)\, T_\alpha(\varphi(\varsigma)) - \varphi(\varsigma) T_\alpha(\phi(\varsigma))}{g^2(\varsigma)},$
4. $T_\alpha(\varphi(\varsigma))$ =0 if $\varphi(\varsigma) = \lambda,\ \lambda \in \mathcal{R}$.
5. If $\varphi(\varsigma)$ is a differentiable function with respect to ς, then, $T_\alpha(\varphi)(\varsigma) = \varsigma^{\,1-\alpha} \frac{d\,\varphi(\varsigma)}{d\varsigma}$.
6. If $\varphi(\varsigma)$ is an (n+1)-differentiable function with respect to ς, then, $T_\alpha\big(\varphi(\varsigma)\big) = \varsigma^{n+1-\alpha}\,\frac{d\varphi(\varsigma)}{d\varsigma}$, where $\alpha \in [n-1, n)$.

Proposed Analytical Method

The investigation of the proposed approach is dealt with in this section.

Theorem 1

The following is a property of the second-order α-fractional derivative, $T_{2\alpha}$, for the real function $\varphi: [a, \infty) \to \mathcal{R}$ in the domain I_α:

$$T_{2\alpha}\big(\varphi(\varsigma)\big) = \zeta^{1-2\alpha}\,\big(\varsigma\varphi''(\varsigma) + (1-\alpha)\,\varphi'(\varsigma)\big). \tag{8}$$

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Proof

From the property $T_\alpha(\varphi)(\varsigma) = \varsigma^{1-\alpha}\frac{d\,\varphi(\varsigma)}{d\varsigma}$, utilizing the α-fractional derivative of its two sides, yields the following:

$$T_{2\alpha}(\varphi(\varsigma)) = \varphi^{(2\alpha)}(\varsigma) = T_\alpha(T_\alpha(\varphi(\varsigma))) = T_\alpha(\zeta^{1-\alpha}\,\varphi'(\varsigma))$$
$$= \varsigma^{1-\alpha}\,T_\alpha(\varphi'(\varsigma)) + \varphi'(\varsigma)\,T_\alpha(\varphi(\varsigma^{1-\alpha})).$$

Now, the property and Equation (8) provide two transformations that may be applied to transform FDE to ODE of first- or second-order, respectively, using the properties of the α-fractional derivative.

Algorithms of the Proposed Analytical Method for Solving Second-Order ODEs

To solve a second-order FDE, the following steps should be followed:

Step I: We utilize the transformation in Equations (8) to solve the FDE in Equation (4) with ICs in Equation (5), yielding the following ODE.

$$\varphi(\varsigma, w(\varsigma), w'(\varsigma), \zeta^{1-\alpha}\,w'(\varsigma), \varsigma^{1-2\alpha}\,(\zeta w''(\varsigma) + (1-\alpha)\,w'(\varsigma)) = 0 \qquad (9)$$

Step II: Use an appropriate analytical technique to solve Equation (9), and then use Equation (5)'s IC to obtain the general solution of Equation (9). This solution, therefore, becomes the same as the FDE solution in Equation (4) with ICs in Equation (5).

Classical Runge-Kutta-Nystrom Numerical Method

In this subsection, the Runge-Kutta-Nystrom (RKN) method has been introduced for solving ODEs; then, the RKN technique has been used to solve the dual of second-order quasi-linear FDE in Equation (4) with ICs in Equation (5). The formula of the RKN method is given as follows:

$$\upsilon_{n+1} = \upsilon_n + h\,\upsilon'_n + h^2\sum_{i=1}^{s} b_i k_i,$$

$$\upsilon'_{n+1} = \upsilon'_n + h\sum_{i=1}^{s} b'_i\,k_i\,,$$

where,

$$k_i = f(x_n + c_i h, \upsilon_n + c_i h\;\upsilon'_n + h^2\sum_{j=1}^{s} a_{ij}k_j,$$

As well as, $h = \frac{x_m - x_0}{m}$ and $x_n = a + nh$; for $n = 0,1,2,\ldots,m$.

Implementation

The effectiveness of the proposed techniques has been shown in this section via the implementation of several cases.

Example 1: Consider the following FDE:

$$\sqrt{\varsigma}\,y^{\frac{3}{2}}(\varsigma) = \varsigma\,y''(\varsigma) + 0.25\,y'(\varsigma) + \sqrt{\varsigma}\,(y''(\varsigma) + y'(\varsigma) - 2\varsigma - 2);\;\; \varsigma > 0, \qquad (10)$$

with the ICs.: $y(0) = 1, y^{\frac{3}{4}}(0) = 0$. Using the relation in Equation (8), then, FDE in Equation (10) converts to the following ODE $y''(\varsigma) + y'(\varsigma) = 2\varsigma + 1,\; \varsigma > 0$, where the ICs are $y(0) = 1, y'(0) = 0$. This initial value problem (IVP) has the exact solution $y(\varsigma) = \varsigma^2 + 1$ hence, it satisfies the IVP of FDE in Equation (10).

Example 2: Consider the following FDE:

$$\sqrt[7]{\varsigma}\,y^{\frac{4}{7}}(\varsigma) = \varsigma\,y''(\varsigma) + \frac{3}{7}\,y'(\varsigma) + \sqrt[7]{\varsigma}(y''(\varsigma) + y(\varsigma) - \varsigma^2 - 5), \qquad \varsigma > 0, \qquad (11)$$

with the ICs.: $y(0) = 3, y^{\frac{3}{4}}(0) = 0$. Using the relation in Equation (8), then, FDE in Equation (11) converts to the following ODE$y''(\varsigma) + y(\varsigma) = \varsigma^2 + 5,\; \varsigma > 0$, where the ICs are $y(0) = 3, y'(0) = 0$. This IVP has the exact solution $y(\varsigma) = \varsigma^2 + 3$ hence, it satisfies the IVP of FDE in Equation (11).

Example 3: Consider the following FDE:

$$\sqrt[3]{\varsigma^2}\,y^{\frac{5}{6}}(\varsigma) = \varsigma\,y''(\varsigma) + \frac{1}{6}\,y'(\varsigma) + \sqrt[3]{\varsigma^2}(y''(\varsigma) - y(\varsigma)), \qquad \varsigma > 0, \qquad (12)$$

with the ICs.: $y(0) = 1, y^{\frac{3}{4}}(0) = 0$. Using the relation in Equation (8), then, FDE in Equation (12) converts to the following ODE $y''(\varsigma) = y(\varsigma)$, where the ICs are $y(0) = 1, y'(0) = -1$. This IVP has the exact solution $y(\varsigma) = e^{-\varsigma}$ hence, it satisfies the IVP of FDE in Equation (12).

Example 4: Consider the FDE that follows:

$$-\sqrt[11]{\varsigma}\,y^{\frac{6}{11}}(\varsigma) + \varsigma\,y''(\varsigma) + \sqrt[11]{\varsigma}(y''(\varsigma) + y(\varsigma) - 2e^{-\varsigma}) = 0, \qquad \varsigma > 0, \qquad (13)$$

with the ICs.: $y(0) = 1, y^{\frac{3}{4}}(0) = 0$. Using the relation in Equation (8), then, FDE in Equation (13) converts to the following ODE $y''(\varsigma) + y(\varsigma) = 2e^{-\varsigma}, > 0$, where the ICs are $y(0) = 1, y'(0) = -1$. This IVP has the exact solution $y(\varsigma) = e^{-\varsigma}$ hence, it satisfies the IVP of FDE in Equation (13).

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Example 5: Consider the following FDE that follows:

$$-\sqrt[6]{\varsigma}\, y^{\frac{7}{12}}(\varsigma)_{\varsigma} y''(\varsigma) + \frac{5}{12}\, y'(\varsigma) + \sqrt[6]{\varsigma}\, \left(y''(\varsigma) + \left(y'(\varsigma)\right)^{4} - e^{-\varsigma^{2}} - 16\, \varsigma^{2} e^{-4\varsigma^{2}}\right) = 0, \qquad \varsigma > 0, \tag{14}$$

with the ICs.: $y(0) = 1, y^{\frac{3}{4}}(0) = 0$. Using the relation in Equation (8), then, FDE in Equation (14) converts to the following ODE $y''(\varsigma) + \left(y'(\varsigma)\right)^{4} - e^{-\varsigma^{2}} - 16\, \varsigma^{2} e^{-4\varsigma^{2}} = 0,\ c > 0$, where the ICs are $y(0) = 1, y'(0) = 0$. This IVP satisfies the IVP of FDE in Equation (14). The analytical solution to Equation (14) is complicated. Consequently, the RKN approach is used to compute both solutions of the ODE in Equation (14), and the results are displayed in Figure 1.

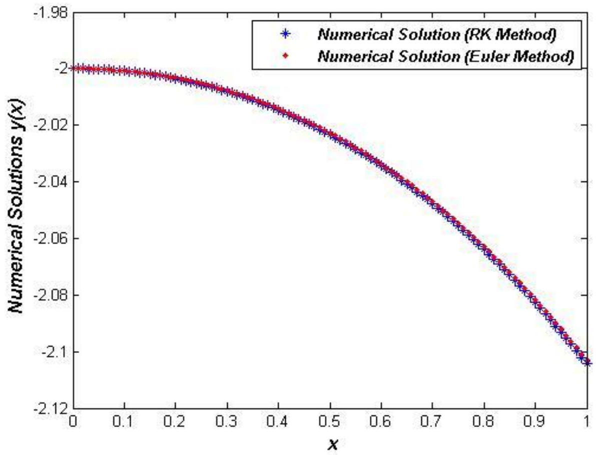

Figure 1. Numerical Solution (Euler and RKN Methods) for Example 5

Discussion

This study introduced and applied the newly defined α-transform fractional derivative to address the challenges of solving second-order fractional differential equations (FDEs). The α-transform offers a novel mechanism for converting FDEs into ordinary differential equations (ODEs), thereby enabling the use of established analytical and numerical techniques for solution. This transformation is particularly valuable because it simplifies the mathematical complexity typically associated with fractional calculus, making the equations more tractable for both theoretical analysis and computational implementation.

The efficiency and practicality of the α-transform were demonstrated through five illustrative examples. In Examples 1 through 4, the transformed ODEs were solved analytically, yielding exact solutions that validate the accuracy of the method. Example 5, which involved a more complex system, was solved numerically using a modified Runge–Kutta–Nyström (RKN) technique. This adaptation of the RKN method to fractional systems further highlights the flexibility of the α-transform in accommodating both analytical and numerical approaches.

The results across all examples consistently showed that the α-transform not only preserves the essential dynamics of the original FDEs but also enhances computational efficiency. The method proved particularly effective in reducing solution time and improving stability, which are critical factors in real-world applications of fractional models. Moreover, the ability to integrate the α-transform with existing numerical schemes like RKN opens new avenues for solving higher-order and nonlinear fractional systems.

These findings suggest that the α-transform could serve as a foundational tool in fractional modeling, especially in fields such as physics, engineering, and finance, where fractional dynamics are increasingly used to describe complex systems. Future research may explore its application to multi-dimensional systems, stochastic fractional models, and real-time simulations.

Conclusion

In summary, the α-transform fractional derivative presents a promising approach for simplifying and solving second-order FDEs. By converting these equations into ODEs, the method enables both analytical and numerical solutions with improved accuracy and efficiency. The successful implementation across five examples—four solved analytically and one numerically—demonstrates the robustness and versatility of the proposed technique. This work lays the groundwork for broader applications of the α-transform in fractional calculus and computational modeling.

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Conflict of Interest
The authors declare that they have no conflicts of interest.

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